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Formulation of a Well-Posed Stokes-Brinkman Problem with a Permeability Tensor

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***Abstract*—We consider a slow flow with incompressible viscous fluid flowing through two different domains: a porous medium and adjacent free-fluid region. With the slow flow problem the Stokes equation is employed in this study. To match the shear stress at free-fluid/porous-medium interface and to have a flexibility to make choice of boundary conditions at the interface, we apply Brinkman equations in the porous medium domain. A mixed finite element method is used to discretize the model to obtain a weak Stokes-Brinkman formulation. We establish the continuity of the bilinear form and then provide the well-posedness of the discrete problem of the Stokes-Brinkman equation when permeability coefficient is considered to be an n-dimensional tensor. This result can also be applied to a free boundary problem as long as the boundary conditions at the interface is in the Sobolev space $H^{1/2}(\partial\Omega)$.**

***Keywords*— Well-posedness; Stokes-Brinkman; Permeability tensor; Moving solid phases; Finite element; Porous media**

I. Introduction

The approximation of velocity of fluid flow through a porous medium and adjacent free fluid region is important in several applications such as fluid flow through natural rice field [1] which is one of classical examples that fluid is moved by a pressure gradient. In this research we consider a model that fluid is moved by self-propelled solid phases such as animal hair. The configuration showing the geometry of our model is illustrated in Figure 1. It displays an ideal cell of moving solid phases in domain Ω_1 and free-fluid region Ω_2 resides above Ω_1.

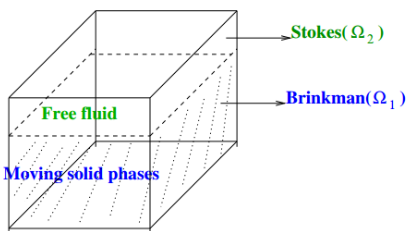

Fig. 1. A snapshot of a cell of a free-fluid region resides above the porous medium.

A model using an upscaling technique is employed so that we do not have to consider the motion of each individual moving solid phases but rather what all solid phases do collectively and can be viewed as a porous medium with the self-propelled solid phase. We employ the coupled Stokes-Brinkman system [2, 3]:

$$\mu\mathbf{k}^{-1}\cdot\left(\varepsilon^l\mathbf{v}^l\right)+\nabla p-\frac{\mu}{\varepsilon^l}\Delta\left(\varepsilon^l\mathbf{v}^l\right)=\rho\mathbf{g}+\mu\mathbf{k}^{-1}\cdot\varepsilon^l\mathbf{v}^s+\frac{\mu}{\varepsilon^l}\nabla f\ ,\quad(1)$$

$$\nabla\cdot\left(\varepsilon^l\mathbf{v}^l\right)=f,\quad(2)$$

where $f=-\dot{\varepsilon}^l/\left(1-\varepsilon^l\right)+\nabla\cdot\varepsilon^l\mathbf{v}^s$; μ is a dynamic viscosity; $\mathbf{k}^{-1}$ is the inverse of the permeability tensor; ε^l is the porosity; $\mathbf{v}^l$ and $\mathbf{v}^s$ are the velocities of the liquid and solid phases, respectively; p is the pressure; $\mathbf{d}^l=0.5\left(\nabla\mathbf{v}^l+\left(\nabla\mathbf{v}^l\right)^T\right)$ is the rate of deformation tensor; ρ is the fluid density; $\mathbf{g}$ is the gravity; $\dot{\varepsilon}^l$ is the material time derivative of the porosity with respect to the solid phase, $\dot{\varepsilon}^l=\partial\varepsilon^l/\partial t+\mathbf{v}^s\cdot\nabla\varepsilon^l$. The introduction of an effective viscosity parameter, μ/ε^l, in the additional term of Darcy's law within the Brinkman equation allows the matching of the stress between the two domains. Without the inverse of the permeability term, $\mu\mathbf{k}^{-1}\cdot\left(\varepsilon^l\mathbf{v}^l-\varepsilon^l\mathbf{v}^s\right)$ and the source term f, the Brinkman equation becomes the Stokes equation. With divergence-free condition, this is the case in Ω_2 while the Brinkman equation with the nondivergence form (2) is used in Ω_1. The system of equations is derived from Hybrid Mixture Theory (HMT) [4, 5], which is an upscaling method. The derivation of the momentum equation can be found in [2] while the derivation of the conservation of mass, equation (2), used in this problem is in [3, 5]. In this work, we present the existence and uniqueness of the Stokes-Brinkman equations for the numerical problem when the permeability coefficient is an n-dimensional tensor.

Typically, Darcy's law is used with the Beavers-Joseph condition in the porous medium and the Stokes equation is used in free-fluid region for slow flow problem [6, 7, 8]. To match the shear stress at the free-fluid/porous-medium interface, in this study, we use the Stokes-Brinkman equation, cf. e.g., [9, 10]. The Stokes-Brinkman equation have been studied by several authors in many aspects such as calculating the numerical solutions [11, 12, 13] or analytically finding drag force from the equations [14], comparing with Stokes-Darcy in both theoretical and numerical aspects [15, 16, 17, 18] and proving the well-posedness of the equations with different boundary condition [19, 20]. For example, Chen et al. [15] analytically compared results of Stokes-Brinkman and Stokes-Darcy's equations with Beavers-Joseph interface condition in 1-dimensional and quasi-2-dimensional cases and also considered the coupling of the Stokes and Darcy systems with different choices for the interface conditions. Angot [19] studied the Stokes-Brinkman equation with jump embedded boundary conditions on an immersed interface. He showed the well-posedness of the system of equations with Ochoa-Tapia & Whitaker (1995) interface conditions and Stokes-Darcy with Beavers & Joseph (1967) conditions. Ingram [20] analyzed a finite element discretization of the Brinkman equation for modeling non-Darcian fluid flow with different boundary conditions. He proved the well-posedness of the problem. He also established the existence and uniqueness of the solutions for steady Navier-Stokes equation. However, all of the previous works had been studied for static solid phases of the Stokes-Brinkman equations with a constant permeability. In this study, we provide the well-posedness of the discrete problem of the Stokes-Brinkman equations when fluid flows through the domain $\Omega_1 \cup \Omega_2$ by the movement of the solid phases and the permeability coefficient is an n-dimensional tensor. This model can be applied to more realistic problems, which previously this had been shown only for the scalar coefficient in [20] and second-order tensor in [21].

In Section 2, we derive the weak form of the Stokes-Brinkman equations using a mixed finite element method. Theorems and the definition of a dual operator are provided in Section 3. The well-posedness of the weak form of the Stokes-Brinkman system of equations is shown in Section 4 for a general n-dimensional permeability tensor. The conclusion is drawn in Section 5.

II. Weak Stokes-Brinkman Equations

To prove the continuity and coercivity of the bilinear form $a(\cdot,\cdot)$ and then the existence and uniqueness of the discrete form of the Stokes-Brinkman equations, we first find a weak form of the equations and begin with introducing some notations and spaces which are employed from [21] defined as follows.

$$L_0^2(\Omega) = \left\{ q \in L^2(\Omega) : \int_\Omega q \, d\Omega = 0 \right\}, \tag{3}$$

$$H_0^1(\Omega) = \left\{ \mathbf{w} \in H^1(\Omega) : \mathbf{w}|_{\partial\Omega} = \mathbf{0} \right\}, \tag{4}$$

$$H_s^1(\Omega) = \left\{ \mathbf{w} \in H^1(\Omega) : \mathbf{w}|_{\partial\Omega} = \mathbf{s} \right\}, \tag{5}$$

$$H^{-1}(\Omega) = \left(H_0^1(\Omega) \right)', \text{ the dual of } H_0^1(\Omega), \tag{6}$$

$$V = \left\{ \mathbf{w} \in H^1(\Omega) : \mathbf{w}|_{\partial\Omega} = 0 \text{ and } \nabla\cdot\mathbf{w} = 0 \right\}, \tag{7}$$

$$V^\perp = \left\{ \mathbf{w}^\perp \in H_0^1(\Omega) : \int_\Omega \mathbf{w}^\perp \cdot \mathbf{w} = 0 \ \forall \mathbf{w} \in V \right\}, \tag{8}$$

$$V^0 = \left\{ \mathbf{w}' \in H^{-1}(\Omega) : \langle \mathbf{w}', \mathbf{w} \rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)} = 0 \ \forall \mathbf{w} \in V \right\}, \tag{9}$$

where $V^\perp$ denotes the orthogonal of V in $H_0^1(\Omega)$ associated with the $H^1(\Omega)$ seminorm $|\cdot|_{H^1(\Omega)}$; V^0 is the polar set of V; $\langle \cdot,\cdot \rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)}$ represents the duality pairing, in particular, between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$, and the existence of the function $\mathbf{s} \in H^{1/2}(\partial\Omega)$ used in the definition of seminorm is ensured by Trace Theorem 2, below. Note that for a n-dimensional domain, $\mathbf{w} \in H^1(\Omega)^n$ and $\nabla\mathbf{w} \in H^1(\Omega)^{n\times n}$. However, for simplicity, we write $\mathbf{w} \in H^1(\Omega)$ in either case and the meaning follows from the context. Recall the Brinkman and continuity equations,

$$\mu\mathbf{k}^{-1}\cdot\left(\varepsilon^l \mathbf{v}^l\right) + \nabla p - \frac{\mu}{\varepsilon^l}\Delta\left(\varepsilon^l \mathbf{v}^l\right) = \rho\mathbf{g} + \mu\mathbf{k}^{-1}\cdot\varepsilon^l\mathbf{v}^s + \frac{\mu}{\varepsilon^l}\nabla f, \tag{10}$$

$$\nabla\cdot\left(\varepsilon^l\mathbf{v}^l\right) = f, \tag{11}$$

where the velocity of the liquid $\mathbf{v}^l$ and the pressure p are unknown. Let the vector $\mathbf{f}_1 = \rho\mathbf{g} + \mu\mathbf{k}^{-1}\cdot\varepsilon^l\mathbf{v}^s \in H^{-1}(\Omega)$ with the following norm:

$$\|\mathbf{f}_1\|_{H^{-1}(\Omega)} = \sup_{\mathbf{w}\in H_0^1(\Omega), \mathbf{w}\neq 0} \frac{\langle \mathbf{f}_1, \mathbf{w} \rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)}}{\|\mathbf{w}\|_{H^1(\Omega)}}, \tag{12}$$

where $\mathbf{v}^s$ is a bounded continuous function and $\|\cdot\|_{H^1(\Omega)}$ represents the standard norm for $H^1(\Omega)$. Assume ε^l is fixed in space and define

$$\mathbf{v} = \varepsilon^l\mathbf{v}^l \text{ and } \mathbf{f} = \mathbf{f}_1 + \left(\mu/\varepsilon^l\right)\nabla f. \tag{13}$$

We obtain the Stokes-Brinkman equations in the following form

$$\mu\mathbf{k}^{-1}\cdot\mathbf{v} + \nabla p - \frac{\mu}{\varepsilon^l}\Delta\mathbf{v} = \mathbf{f}, \tag{14}$$

$$\nabla\cdot\mathbf{v} = f. \tag{15}$$

Define the linear and bilinear functionals

$$a(\mathbf{v},\mathbf{w}) = \int_\Omega \frac{\mu}{\varepsilon^l}\nabla\mathbf{v} : \nabla\mathbf{w} + \int_\Omega \mu\left(\mathbf{k}^{-1}\cdot\mathbf{v}\right)\cdot\mathbf{w}, \tag{16}$$

$$b(\mathbf{v},q) = -\int_\Omega q\nabla\cdot\mathbf{v}, \tag{17}$$

$$c_1(\mathbf{w}) = \langle \mathbf{f}_1,\mathbf{w}\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)} - \int_\Omega \frac{\mu}{\varepsilon^l} f\nabla\cdot\mathbf{w}, \tag{18}$$

$$c_2(q) = -\int_\Omega fq. \tag{19}$$

Then, the weak formulation of (14) and (15) can be expressed as follows.

Problem 1. *(Weak Stokes-Brinkman) Find* $\mathbf{v}\in H_s^1(\Omega)$ *and* $p\in L_0^2(\Omega)$ *such that*

$$\forall\mathbf{w}\in H_0^1(\Omega),\quad a(\mathbf{v},\mathbf{w}) + b(\mathbf{w},\mathrm{p}) = c_1(\mathbf{w}), \tag{20}$$

$$\forall q\in L_0^2(\Omega),\quad b(\mathbf{v},p) = c_2(q). \tag{21}$$

III. Preliminary definition and theorems

In this section, we present selected definitions, lemmas and theorems required to use in the proof of the well-posedness of the weak form of our model to make this paper self-contained though they are provided in several places such as [20] and [21] while the definitions of Sobolev norm, seminorm and weak derivative are based on [22]. We first introduce the direct and inverse trace theorem for $H^1(\Omega)$ as follows [23].

Theorem 2. *(Direct and Inverse Trace Theorem for* $H^1(\Omega)$*) There exist positive constants* K *and* K' *such that, for each* $\mathbf{w}\in H^1(\Omega)$, *its trace on* $\partial\Omega$ *belongs to* $H^{1/2}(\partial\Omega)$ *and* $\|\mathbf{w}\|_{H^{1/2}(\partial\Omega)} \le K\|\mathbf{w}\|_{H^1(\Omega)}$. *Conversely, for each given function* $\mathbf{s}\in H^{1/2}(\partial\Omega)$, *there exists a function* $\mathbf{v}_\mathbf{s}\in H^1(\Omega)$ *such that its trace on* $\partial\Omega$ *coincides with* $\mathbf{s}$ *and*

$$\|\mathbf{v}_\mathbf{s}\|_{H^1(\Omega)} \le K'\|\mathbf{s}\|_{H^{1/2}(\partial\Omega)}. \tag{22}$$

This theorem ensures that if $\mathbf{s}\in H^{1/2}(\partial\Omega)$, then there exists $\mathbf{v}_\mathbf{s}\in H^1(\Omega)$ such that the trace of $\mathbf{v}_\mathbf{s}$ on $\partial\Omega$ is $\mathbf{s}$. The following formulation will be used in the proof of the Theorem 8.

Theorem 3. $\exists!\mathbf{v}_0\in V^\perp\subset H_0^1(\Omega)$ *such that* $\nabla\cdot\mathbf{v}_0 = f - \nabla\cdot\mathbf{v}_\mathbf{s}$.
Proof. The proof of this theorem is provided in [21].

Next theorem states that the divergence operator is an isomorphism between $L_0^2(\Omega)$ and $V^\perp$, and the Ladyzhenskaya-Babuška-Brezzi (LBB) condition, which is required for the stability of a mixed finite element method, is mentioned [24, 25].

Theorem 4. *Let* Ω *be connected. Then*
1) the operator ***grad*** *is an isomorphism of* $L_0^2(\Omega)$ *onto* V^0,
2) the operator ***div*** *is an isomorphism of* $V^\perp$ *onto* $L_0^2(\Omega)$.
Moreover, there exists $\beta>0$ *such that*

$$\inf_{q\in L_0^2(\Omega)}\sup_{\mathbf{w}\in H_0^1(\Omega)} \frac{b(\mathbf{w},q)}{\|\mathbf{w}\|_{H^1(\Omega)}\|q\|_{L^2(\Omega)}} \ge \beta > 0 \tag{23}$$

and for any $q\in L_0^2(\Omega)$, *there exists a unique* $\mathbf{v}\in V^\perp\subset H_0^1(\Omega)$ *satisfying*

$$\|\mathbf{v}\|_{H^1(\Omega)} \le \beta^{-1}\|q\|_{L^2(\Omega)}. \tag{24}$$

Note that the equation (23) is known as the LBB condition [25]. We next state the definition of linear operators and their dual operator and then rewrite the Problem 1 in the form of the linear operator. This simple change allows us to prove the existence and uniqueness of the pressure term. Recall that the dual spaces of $L_0^2(\Omega)$ and $H^{-1}(\Omega)$ are $\left(L_0^2(\Omega)\right)'$ and $\left(H_0^1(\Omega)\right)'$, respectively, i.e., $\left(L_0^2(\Omega)\right)' = L_0^2(\Omega)$ and $\left(H_0^1(\Omega)\right)' = H^{-1}(\Omega)$.

Definition 5. *Let* $\mathbf{v},\mathbf{w}\in H^1(\Omega)$ *and* $q\in L_0^2(\Omega)$. *Define linear operators* $A: H_0^1(\Omega)\to H^{-1}(\Omega)$ *and* $B: H_0^1(\Omega)\to L_0^2(\Omega)$ *by*

$$\langle A\mathbf{v},\mathbf{w}\rangle_{H_0^1(\Omega)\times H^{-1}(\Omega)} := a(\mathbf{v},\mathbf{w}),\qquad \forall\mathbf{v},\mathbf{w}\in H_0^1(\Omega) \tag{25}$$

$$\langle B\mathbf{v},q\rangle_{H_0^1(\Omega)\times L_0^2(\Omega)} := b(\mathbf{v},q),\ \forall\mathbf{v}\in H_0^1(\Omega),\forall q\in L_0^2(\Omega). \tag{26}$$

Let $B'\in\mathcal{L}\left(L_0^2(\Omega);H^{-1}(\Omega)\right)$ *be the dual operator of* B, i.e.

$$\langle B'q,\mathbf{v}\rangle = \langle q,B\mathbf{v}\rangle := b(\mathbf{v},q),\ \forall\mathbf{v}\in H_0^1(\Omega),\forall q\in L_0^2(\Omega). \tag{27}$$

With these operators, Problem 1 is equivalently written in the form:

Problem 6. *Find* $\mathbf{v}\in H_s^1(\Omega)$, $p\in L_0^2(\Omega)$ *such that*

$$A\mathbf{v} + B'p = \mathbf{f}\quad \text{in } H^{-1}(\Omega) \tag{28}$$

$$B\mathbf{v} = f\quad \text{in } L_0^2(\Omega). \tag{29}$$

IV. WELL-POSEDNESS OF THE STOKES-BRINKMAN PROBLEM

Even though Ingram [20] and [21] proved the well-posedness of the Stokes-Brinkman problem, it was for only with a constant and second-tensor permeability, respectively. Here we generalize the result of the discrete equations when $\mathbf{k}$ is an n-dimensional tensor. To present the well-posedness of the Stokes-Brinkman equations, we first show that the linear and bilinear functionals (16)-(19) are continuous and $a(\cdot,\cdot)$ is coercive. Then we use these properties to show the existence and uniqueness of the equations in Theorem 8 below.

Theorem 7. *The linear functionals $c_1(\mathbf{w}), c_2(q)$ and bilinear functionals $a(\cdot,\cdot)$, $b(\cdot,\cdot)$ are continuous and $a(\cdot,\cdot)$ is coercive, i.e.,*

$$a(\mathbf{w},\mathbf{w}) \geq C_c \|\mathbf{w}\|^2_{H^1(\Omega)} \qquad (30)$$

where $C_c = \min\{\mu/\varepsilon, \mu C_k\}$; C_k is a positive number. In particular,

$$c_1(\mathbf{w}) \leq \left(\|\mathbf{f_1}\|_{H^{-1}(\Omega)} + \sqrt{n}\frac{\mu}{\varepsilon}\|f\|_{L^2(\Omega)} \right) \|\mathbf{w}\|_{H^1(\Omega)}, \quad \forall \mathbf{w} \in H^1(\Omega), \qquad (31)$$

$$c_2(q) \leq \|f\|_{L^2(\Omega)} \|q\|_{L^2(\Omega)}, \quad \forall q \in L^2(\Omega), \qquad (32)$$

$$b(\mathbf{v},q) \leq \sqrt{n}|\mathbf{v}|_{H^1(\Omega)} \|q\|_{L^2(\Omega)}, \quad \forall \mathbf{v} \in H^1(\Omega), \quad \forall q \in L^2(\Omega), \qquad (33)$$

$$a(\mathbf{v},\mathbf{w}) \leq C_a \|\mathbf{v}\|_{H^1(\Omega)} \|\mathbf{w}\|_{H^1(\Omega)}, \quad \forall \mathbf{v} \in H^1(\Omega), \quad \forall \mathbf{w} \in H^1(\Omega), \qquad (34)$$

where n is the dimensional number and $C_a = \max\{\mu/\varepsilon^l, n\mu \max_{1\leq i,j\leq n}|k_{ij}^{-1}|\}$.

Proof. Since the linearity of $c_1(\mathbf{w})$ and $c_2(q)$ and bilinearity of $a(\mathbf{v},\mathbf{w})$ and $b(\mathbf{v},q)$ are obvious and the continuities of $c_1(\mathbf{w})$, $c_2(q)$ and $b(\mathbf{v},q)$ have been shown in [21], we next show $a(\mathbf{v},\mathbf{w})$ is continuous. To prove the continuity of $a(\mathbf{v},\mathbf{w})$ for an n-dimensional domain, we first introduce Young's inequality: $ab \leq a^p/p + b^q/q$ where $a,b \geq 0$, $p,q > 0$ and $(1/p)+(1/q)=1$ which is often used below, write vectors $\mathbf{v}$ and $\mathbf{w}$ in component forms: $\mathbf{v}=(v_1,v_2,...,v_n)$ and $\mathbf{w}=(w_1,w_2,...,w_n)$ and consider

$$\begin{aligned}
\|\mathbf{k^{-1}}\cdot\mathbf{v}\|^2_{L^2(\Omega)} &= \sum_{i=1}^n \int_\Omega \left(\sum_{j=1}^n k_{ij}^{-1} v_j \right)^2 \\
&\leq \sum_{i=1}^n \left(\int_\Omega \left(\sum_{j=1}^n \left(k_{ij}^{-1}v_j\right)^2 + 2\sum_{j=1}^{n-1}\sum_{k>j}^{n} \left|k_{ij}^{-1}k_{ik}^{-1}v_jv_k\right| d\Omega \right) \right) \\
&\leq n\left(\max_{1\leq i,j\leq n} \left|k_{ij}^{-1}\right|^2 \right) \int_\Omega \left(\sum_{k=1}^n v_k^2 + 2\sum_{j=1}^{n-1}\sum_{k>j}^{n} \left|v_jv_k\right| \right) d\Omega \\
&\leq n\left(\max_{1\leq i,j\leq n} \left|k_{ij}^{-1}\right|^2 \right) \\
&\quad \int_\Omega \left(\sum_{k=1}^n v_k^2 + \sum_{j=1}^{n-1}\sum_{k>j}^{n} \left(\left|v_j\right|^2 + \left|v_k\right|^2 \right) \right) d\Omega \\
&\leq n^2 \left(\max_{1\leq i,j\leq n} \left|k_{ij}^{-1}\right|^2 \right) \int_\Omega \sum_{k=1}^n v_k^2 d\Omega \\
&= n^2 \left(\max_{1\leq i,j\leq n} \left|k_{ij}^{-1}\right|^2 \right) \|\mathbf{v}\|^2_{L^2(\Omega)},
\end{aligned} \qquad (35)$$

where Young's inequality is applied to the third inequality. We then employ the inequality (35) to complete the proof of the continuity of $a(\mathbf{v},\mathbf{w})$ as follows.

$$\begin{aligned}
|a(\mathbf{v},\mathbf{w})| &= \left| \int_\Omega \frac{\mu}{\varepsilon^l}\nabla\mathbf{v}:\nabla\mathbf{w} + \int_\Omega \mu\left(\mathbf{k^{-1}}\cdot\mathbf{v}\right)\cdot\mathbf{w} \right| \\
&\leq \left| \int_\Omega \frac{\mu}{\varepsilon^l}\nabla\mathbf{v}:\nabla\mathbf{w} \right| + \left| \int_\Omega \mu\left(\mathbf{k^{-1}}\cdot\mathbf{v}\right)\cdot\mathbf{w} \right| \\
&\leq \frac{\mu}{\varepsilon^l}\|\nabla\mathbf{v}\|_{L^2(\Omega)}\|\nabla\mathbf{w}\|_{L^2(\Omega)} + \mu\|\mathbf{k^{-1}}\cdot\mathbf{v}\|_{L^2(\Omega)}\|\mathbf{w}\|_{L^2(\Omega)} \\
&\leq \frac{\mu}{\varepsilon^l}\|\nabla\mathbf{v}\|_{L^2(\Omega)}\|\nabla\mathbf{w}\|_{L^2(\Omega)} \\
&\quad + n\mu \max_{1\leq i,j\leq n}\left|k_{ij}^{-1}\right| \|\mathbf{v}\|_{L^2(\Omega)}\|\mathbf{w}\|_{L^2(\Omega)} \\
&\leq C_a \|\mathbf{v}\|_{H^1(\Omega)}\|\mathbf{w}\|_{H^1(\Omega)},
\end{aligned}$$

where $C_a = \max\left\{\mu/\varepsilon^l, n\mu \max_{1\leq i,j\leq n}\left|k_{ij}^{-1}\right|\right\}$ and (35) is employed to the third inequality.

Before proving the coercivity of the bilinear form $a(\mathbf{w},\mathbf{w})$, we first consider, for n-dimensions,

$$\left(\mathbf{k^{-1}}\cdot\mathbf{w}\right)\cdot\mathbf{w} = \sum_{i=1}^n k_{ii}^{-1}w_i^2 + 2\sum_{i=1}^{n-1}\sum_{j>i}^{n} k_{ij}^{-1}w_iw_j. \qquad (36)$$

Since in our problem $\mathbf{k^{-1}}$ is positive definite, the diagonal entries of $\mathbf{k^{-1}}$ are positive numbers, which imply that the first term of the right-hand side of (36) is a positive number. We now focus on the second term $2\sum_{i=1}^{n-1}\sum_{j>i}^{n} k_{ij}^{-1}w_iw_j$. Note that if $k_{ij}^{-1}w_iw_j \geq 0$ for all i and j, it's easy to see that $\left(\mathbf{k^{-1}}\cdot\mathbf{w}\right)\cdot\mathbf{w} \geq \sum_{i=1}^n k_{ii}^{-1}w_i^2 \geq \left(\min_{1\leq i\leq n} k_{ii}^{-1}\right)\sum_{i=1}^n w_i^2 = C_k\sum_{i=1}^n w_i^2$ where

$C_k = \min_{1\le i\le n} k_{ii}^{-1}$. If there exist r and s such that $k_{rs}^{-1} w_r w_s < 0$, we have two possible cases.

Case 1. $k_{rs}^{-1} > 0$ and $w_r w_s < 0$.

Thus, $2k_{rs}^{-1} w_r w_s = -2k_{rs}^{-1}\left|w_r w_s\right| > -k_{rs}^{-1}\left(w_r^2 + w_s^2\right)$, where Young's inequality is applied at the inequality. Without lose of generality, let $r < s$. Thus,

$$\begin{aligned}
\left(\mathbf{k}^{-1}\cdot\mathbf{w}\right)\cdot\mathbf{w} &= \sum_{i=1}^{n} k_{ii}^{-1} w_i^2 + 2\sum_{i=1}^{n-1}\sum_{j>i}^{n} k_{ij}^{-1} w_i w_j \\
&= \sum_{\substack{i=1\\ i\ne r}}^{n} k_{ii}^{-1} w_i^2 + 2\sum_{\substack{i=1\\ i\ne r}}^{n-1}\sum_{j>i}^{n} k_{ij}^{-1} w_i w_j \\
&\quad + k_{rr}^{-1} w_r^2 + \sum_{i=r+1}^{n} 2k_{ri}^{-1} w_r w_i \\
&> \sum_{\substack{i=1\\ i\ne r}}^{n} k_{ii}^{-1} w_i^2 + 2\sum_{\substack{i=1\\ i\ne r}}^{n-1}\sum_{j>i}^{n} k_{ij}^{-1} w_i w_j \\
&\quad + k_{rr}^{-1} w_r^2 + 2k_{r(r+1)}^{-1} w_r w_{r+1} + \dots \\
&\quad - k_{rs}^{-1}\left(w_r^2 + w_s^2\right) + \dots + 2k_{rn}^{-1} w_r w_n \\
&= \sum_{\substack{i=1\\ i\ne r,s}}^{n} k_{ii}^{-1} w_i^2 + 2\sum_{\substack{i=1\\ i\ne r}}^{n-1}\sum_{j>i}^{n} k_{ij}^{-1} w_i w_j \\
&\quad + \left(k_{rr}^{-1} - k_{rs}^{-1}\right) w_r^2 + \left(k_{ss}^{-1} - k_{rs}^{-1}\right) w_s^2 \\
&\quad + 2k_{r(r+1)}^{-1} w_r w_{r+1} + \dots + 2k_{r(s-1)}^{-1} w_r w_{s-1} \\
&\quad + 2k_{r(s+1)}^{-1} w_r w_{s+1} + \dots + 2k_{rn}^{-1} w_r w_n \\
&\ge C_k \sum_{i=1}^{n} w_i^2,
\end{aligned}$$

where

$C_k = \min\left\{d_k, b_k\right\}$, $b_k = \min\left\{\left(k_{rr}^{-1} - k_{rs}^{-1}\right), \left(k_{ss}^{-1} - k_{rs}^{-1}\right)\right\}$

and $d_k = \min_{\substack{1\le i\le n\\ i\ne r,s}} k_{ii}^{-1}$. Note that $b_k > 0$ because $\mathbf{k}^{-1}$ is a diagonally dominant matrix. Therefore,

$\left(\mathbf{k}^{-1}\cdot\mathbf{w}\right)\cdot\mathbf{w} > C_k \sum_{i=1}^{n} w_i^2$.

Case2. $k_{rs}^{-1} < 0$ and $w_r w_s > 0$.

Since $\mathbf{k}^{-1}$ is bounded, $\exists C_b > 0$ such that $k_{rs}^{-1} = -C_b$.

Thus, $2k_{rs}^{-1} w_r w_s = -2C_b\left|w_r\right|\left|w_s\right| > -C_b\left(w_r^2 + w_s^2\right)$.

Thus,

$$\begin{aligned}
\left(\mathbf{k}^{-1}\cdot\mathbf{w}\right)\cdot\mathbf{w} &= \sum_{i=1}^{n} k_{ii}^{-1} w_i^2 + 2\sum_{i=1}^{n-1}\sum_{j>i}^{n} k_{ij}^{-1} w_i w_j \\
&= \sum_{\substack{i=1\\ i\ne r}}^{n} k_{ii}^{-1} w_i^2 + 2\sum_{\substack{i=1\\ i\ne r}}^{n-1}\sum_{j>i}^{n} k_{ij}^{-1} w_i w_j \\
&\quad + k_{rr}^{-1} w_r^2 + \sum_{i=r+1}^{n} 2k_{ri}^{-1} w_r w_i \\
&> \sum_{\substack{i=1\\ i\ne r}}^{n} k_{ii}^{-1} w_i^2 + 2\sum_{\substack{i=1\\ i\ne r}}^{n-1}\sum_{j>i}^{n} k_{ij}^{-1} w_i w_j + k_{rr}^{-1} w_r^2 \\
&\quad + 2k_{r(r+1)}^{-1} w_r w_{r+1} + \dots - C_b\left(w_r^2 + w_s^2\right) + \dots + 2k_{rn}^{-1} w_r w_n \\
&= \sum_{\substack{i=1\\ i\ne r,s}}^{n} k_{ii}^{-1} w_i^2 + 2\sum_{\substack{i=1\\ i\ne r}}^{n-1}\sum_{j>i}^{n} k_{ij}^{-1} w_i w_j + \left(k_{rr}^{-1} - C_b\right) w_r^2 \\
&\quad + \left(k_{ss}^{-1} - C_b\right) w_s^2 + 2k_{r(r+1)}^{-1} w_r w_{r+1} + \dots + 2k_{r(s-1)}^{-1} w_r w_{s-1} \\
&\quad + 2k_{r(s+1)}^{-1} w_r w_{s+1} + \dots + 2k_{rn}^{-1} w_r w_n \\
&\ge C_k \sum_{i=1}^{n} w_i^2,
\end{aligned}$$

where

$C_k = \min\left\{d_r, b_r\right\}$, $b_r = \min\left\{\left(k_{rr}^{-1} - C_b\right), \left(k_{ss}^{-1} - C_b\right)\right\} > 0$

and $d_k = \min_{\substack{1\le i\le n\\ i\ne r,s}} k_{ii}^{-1}$. Therefore, $\left(\mathbf{k}^{-1}\cdot\mathbf{w}\right)\cdot\mathbf{w} > C_k \sum_{i=1}^{n} w_i^2$.

From all of the cases above, we have

$$\exists C_k > 0, \left(\mathbf{k}^{-1}\cdot\mathbf{w}\right)\cdot\mathbf{w} \ge C_k \sum_{i=1}^{n} w_i^2. \tag{37}$$

Integrating (36) both sides, we obtain

$$\int_\Omega \left(\mathbf{k}^{-1}\cdot\mathbf{w}\right)\cdot\mathbf{w} \ge C_k \int_\Omega \sum_{i=1}^{n} w_i^2 = C_k \left\|\mathbf{w}\right\|_{L^2(\Omega)}^2. \tag{38}$$

Hence, the coercivity of the bilinear form

$$\begin{aligned}
a\left(\mathbf{w},\mathbf{w}\right) &= \int_\Omega \frac{\mu}{\varepsilon^l}\nabla\mathbf{w} : \nabla\mathbf{w} + \int_\Omega \mu\left(\mathbf{k}^{-1}\cdot\mathbf{w}\right)\cdot\mathbf{w} \\
&\ge \frac{\mu}{\varepsilon^l}\left\|\nabla\mathbf{w}\right\|_{L^2(\Omega)}^2 + \mu C_k \left\|\mathbf{w}\right\|_{L^2(\Omega)}^2 \ge C_c \left\|\mathbf{w}\right\|_{H^1(\Omega)}^2,
\end{aligned}$$

where $C_c = \min\left\{\mu/\varepsilon, \mu C_k\right\}$. □

We now ready to prove the existence and uniqueness of the Stokes-Brinkman equations. Though the following theorem have been shown in [20] and [21], we state here in the full form of completeness.

Theorem 8. *(Well-posedness of the Stokes-Brinkman equations) Assume that* $\mathbf{f}_1 \in H^{-1}\left(\Omega\right)$, $\mathbf{f}$, $f \in L^2\left(\Omega\right)$ *and* $\mathbf{s} \in H^{1/2}\left(\partial\Omega\right)$. *There exists a unique* $\mathbf{v} \in H_s^1\left(\Omega\right)$, $p \in L_0^2\left(\Omega\right)$ *satisfying Problem 1, equations (20)-(21). Moreover,*

$$\left\|\mathbf{v}\right\|_{H^1(\Omega)} \le \frac{1}{C_c}\left(\left\|\mathbf{f}_1\right\|_{H^{-1}(\Omega)} + \sqrt{n}\,\frac{\mu}{\varepsilon}\left\|f\right\|_{L^2(\Omega)}\right) + \left(\frac{C_a}{C_c} + 1\right)\left\|\hat{\mathbf{v}}\right\|_{H^1(\Omega)} \tag{39}$$

where $\hat{\mathbf{v}} = \mathbf{v}_\mathbf{s} + \mathbf{v}_0$ *and*

$$\left\|p\right\|_{L^2(\Omega)} \le \frac{1}{\beta}\left(\left\|\mathbf{f}_1\right\|_{H^{-1}(\Omega)} + \sqrt{n}\,\frac{\mu}{\varepsilon^l}\left\|f\right\|_{L^2(\Omega)}\right) + \frac{1}{\beta} C_a \left\|\mathbf{v}\right\|_{H^1(\Omega)}. \tag{40}$$

where β *is the constant in* (23).

Proof. We note that the proof of the inequalities (39) and (40) are provided in [21]. Here we show the existence and uniqueness of the velocity by using Lax-Milgram theorem while the well-posedness of the pressure are shown in the second part by employing Theorem 4 and the definition of linear operators and their dual operators.

Restrict $\mathbf{f}_1 \in H^{-1}(\Omega), \mathbf{f}, f \in L^2(\Omega)$ and $\mathbf{s} \in H^{1/2}(\partial\Omega)$. From Theorem 3, we let $\hat{\mathbf{v}} = \mathbf{v}_s + \mathbf{v}_0$. For any $\mathbf{w} \in V$, let $F(\mathbf{w}) = c_1(\mathbf{w}) - a(\hat{\mathbf{v}},\mathbf{w})$. Since $c_1(\cdot)$ is linear and $a(\cdot,\cdot)$ is bilinear, $F(\cdot)$ is linear. Moreover, the continuities of $c_1(\cdot)$ and $a(\cdot,\cdot)$ imply that $F(\cdot)$ is continuous. By employing Lax-Milgram theorem, there exists a unique $\tilde{\mathbf{v}} \in V \subset H_0^1(\Omega)$ such that $a(\tilde{\mathbf{v}},\mathbf{w}) = F(\mathbf{w})$. Define $\mathbf{v} = \tilde{\mathbf{v}} + \mathbf{v}_s + \mathbf{v}_0$. Because $\mathbf{v}_0 \in V^{\perp} \subset H_0^1(\Omega)$, and $\tilde{\mathbf{v}} \in V \subset H_0^1(\Omega), \mathbf{v}|_{\partial\Omega} = \mathbf{s}$. Since $\tilde{\mathbf{v}} \in V$ and $\nabla \cdot \mathbf{v}_0 = f - \nabla \cdot \mathbf{v}_s, \nabla \cdot \mathbf{v} = f$. We now have $\mathbf{v} \in H_s^1(\Omega)$ satisfying the continuity equation.

Next we show that $\mathbf{v}$ is unique. Since $a(\tilde{\mathbf{v}},\mathbf{w}) = F(\mathbf{w}) = c_1(\mathbf{w}) - a(\hat{\mathbf{v}},\mathbf{w})$ and $\mathbf{v} = \tilde{\mathbf{v}} + \hat{\mathbf{v}}$, $a(\mathbf{v},\mathbf{w}) = c_1(\mathbf{w})$. Let $\mathbf{v}_1$ and $\mathbf{v}_2$ satisfy $a(\mathbf{v}_1,\mathbf{w}) = c_1(\mathbf{w})$ and $a(\mathbf{v}_2,\mathbf{w}) = c_1(\mathbf{w})$. Then $a(\mathbf{v}_1 - \mathbf{v}_2,\mathbf{w}) = 0$ for any $\mathbf{w} \in V$. Thus, $a(\mathbf{v}_1 - \mathbf{v}_2, \mathbf{v}_1 - \mathbf{v}_2) = 0.$ Therefore $0 = a(\mathbf{v}_1 - \mathbf{v}_2, \mathbf{v}_1 - \mathbf{v}_2) \geq C_c \|\mathbf{v}_1 - \mathbf{v}_2\|^2_{H^1(\Omega)} \geq 0.$ Since $C_c > 0$, $\|\mathbf{v}_1 - \mathbf{v}_2\|_{H^1(\Omega)} = 0.$ Then $\mathbf{v}_1 = \mathbf{v}_2$ in the $H^1(\Omega)$-norm.

To show that there exists $p \in L_0^2(\Omega)$ satisfying Problem 1 or 6, we define F_1 such that $\langle F_1, \mathbf{w} \rangle = c_1(\mathbf{w})$. Since $a(\tilde{\mathbf{v}},\mathbf{w}) = c_1(\mathbf{w}) - a(\hat{\mathbf{v}},\mathbf{w})$, $c_1(\mathbf{w}) - a(\tilde{\mathbf{v}},\mathbf{w}) - a(\hat{\mathbf{v}},\mathbf{w}) = 0$ or $F_1 - A\tilde{\mathbf{v}} - A\hat{\mathbf{v}} = 0$, in operator notation, $F_1 - A\tilde{\mathbf{v}} - A\hat{\mathbf{v}} \in V^0$. Let $B' = \nabla : L_0^2(\Omega) \to V^0$. From Theorem 4 and the isomorphic property, there exists a unique $p \in L_0^2(\Omega)$ such that $B'p = F_1 - A\tilde{\mathbf{v}} - A\hat{\mathbf{v}} = F_1 - A\mathbf{v}$ or $A\mathbf{v} + B'p = F_1$. □

V. CONCLUSION

In this study, we employ the macroscale Stokes-Brinkman equation for coupled free-fluid/porous-medium viscous flow using Hybrid Mixture Theory and nondimensionalization. The system of equations is established for the porous medium containing moving solid phases such as hairlike structure. We show that the bilinear form $a(\cdot,\cdot)$ is continuous and coercive for n-dimensional permeability coefficient while it is presented in [20] only for a constant coefficient and in [21] for a second-order tensor. We also present the existence and uniqueness of the Stokes-Brinkman system of equations although this is provided in [21]. Numerical solutions of this model using a mixed finite element method will be provided in future work.

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