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Fourier-Hilbert Transform for Certain Space of Generalized Functions

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Abstract. In this paper, we give a generalization of the Fourier-Hilbert transform on a class of Boehmians. Further, we show that the Fourier- Hilbert transform of a distribution is distribution which is analytic in the space of distributions of compact support. Further properties are also obtained .

Keywords- Hilbert transform; Fourier-Hilbert transform; distribution space; Boehmian space.

I. Introduction

Boehmians were first constructed as a generalization of regular Mikusinski operators. The minimal structure necessary for the construction of Boehmians consists of the following elements: (i) A set $\Im$; (ii)A commutative semigroup $(\Re,*)$; (iii) An operation $\star:\Im\times\Re \to \Im$ such that for each $x \in \Im$ and $v_1, v_2, \in \Re, f \star (v_1 * v_2) = (f \star v_1) \star v_2$; (vi) A collection $\Delta \subset \Re^N$ such that: (a) If $f, g \in \Im, (\varepsilon_n) \in \Delta, f \star \varepsilon_n = g \star \varepsilon_n$ for all n, then $f = g$; (b) If $(\varepsilon_n), (\sigma_n) \in \Delta$, then $(\varepsilon_n * \sigma_n) \in \Delta$. Δ is the set of all delta sequences. Consider $A = \{(f_n, \varepsilon_n): f_n \in \Im, (\varepsilon_n) \in \Delta, f_n \star, \varepsilon_m = f_m \star \varepsilon_n, \forall m, n \in N\}$. If $(f_n, \varepsilon_n), (g_n, \sigma_n) \in A$, $f_n \star \sigma_m = (g_m, \varepsilon_n), \forall m, n \in N$, then we say $(f_n \star \sigma_n) \sim (g_n, \varepsilon_n)$. The relation $\sim$ is an equivalence relation in A. The space of equivalence classes in A is denoted by $G(\Im, \Re, \Delta)$. Elements of $G(\Im, \Re, \Delta)$ are general Boehmians.

Between $\Im$ and $G(\Im, \Re, \Delta)$ there is a canonical embedding expressed as $f \to \frac{(f \star, \varepsilon_n)}{(\varepsilon_n)}$.

The operation $\star$ can be extended to $G(\Im, \Re, \Delta)\times \Im\, by\, \frac{(f_n)}{(\varepsilon_n)} \star t. \frac{(f_n \star t)}{(\varepsilon_n)}$. In $G(\Im, \Re, \Delta)$, two type of convergence:

i-A sequence (h_n) in $G(\Im, \Re, \Delta)$ is said to be δ convergent to h in $G(\Im, \Re, \Delta)$, denoted by $h_n \overset{\delta}{\to} h$, if there exists a delta sequence (ε_n) such that $(h_n \star \varepsilon_n)$, $(h \star \varepsilon_n) \in \Im$, $\forall k, n \in N$, and $h_n \star \varepsilon_k \to h \star \varepsilon_k$ as $n \to \infty$, in $\Im$, for every $k \in N$;

ii-A sequence (h_n) in $G(\Im, \Re, \Delta)$ is said to be Δ convergent to h in $G(\Im, \Re, \Delta)$, denoted by $h_n \to h$, if there exists a $(\varepsilon_n) \in \Delta$ such that $(h_n - h)\star \varepsilon_n \in \Im, \forall n \in N$, and $(h_n\text{-}h) \star \varepsilon_n \to 0\ as\ n \to \infty\ in\ \Im$.

The following is equivalent for the statement of δ convergence: $h_n \to h\ (n \to \infty)$ in $G(\Im, \Re, \Delta)$ if and only if there is $f_{n,k}, f_k \in \Im$ and $(\varepsilon_k) \in \Delta$ such that $h_n = \left[\frac{f_{n,k}}{\varepsilon_k}\right], h = \left[\frac{f_k}{\varepsilon_k}\right]$ and for each $k \in N, f_{n,k} \to f_k$ as $n \to \infty\ in\ \Im$.

Several integral transforms were extended to various spaces of generalized functions; namely, distributions [3,16,19], tempered distributions [7], distributions of compact support [16,19], ultradistributions [1,11], tempered ultradistributions and tempered ultraBoehmians [1] and many others.

Recently, many research works are devoted to those inegral transforms that permit a factorization property of Fourier convolution type. Among those integrals we recall here are: Fourier transform, Mellin transform, Laplace transform and some others that have a lot of attraction; the reason this theory becomes an object of study of integral transforms of generalized functions and, hence, of Boehmians.

The Hilbert transform of $f(x)$ via the Fourier transform is defined by

$$f_H(y) := \frac{1}{\pi}\int_0^\infty \big(FI(x)cos(xy) - FR(x)sin(xy)\big)dx \qquad 1$$

where

$$F(y) := \int_{-\infty}^{\infty} f(t)e^{-iyt}dt := FR(y) \quad iFI(y).$$

$FR(y)$ and $FI(y)$ being the real and imaginary components of the Fourier transform of $f(t)$.

The convolution product of two functions is defined as [16]

$$(f * g)(t) = \int_{-\infty}^{\infty} f(x)\, f(x)g(x - t)dy \qquad 2$$

and has a relationship with the Fourier transform with the factorization property

$$F(f * g)(y) = F(f)(y)F(g)(y).$$

II. Fourier-Hilbert Transform of Boehmians

To follow the results of this extension, reader is acquainted to be familiar with the concept of Boehmian spaces. If it were otherwise we refer to $[1-6,8-9,13,15,18]$ for more details.

Let D be the space of test functions of bounded support over R. By delta sequence, we mean a subset of D of sequences $\{\delta_n\}$ such that :

$$\int_{-\infty}^{\infty} \delta_n(x)dx = 1; \quad 3$$

Type equation here.

$$\|\delta_n\| = \int_{-\infty}^{\infty} |\delta_n(x)|\, dx < M, 0 < M \in R; \quad 4$$

and

$$supp\delta_n(x) = 0 \; as \; n \to \infty, \quad 5$$

where $supp\delta_n(x) = \{x \in R: \delta_n(x) \neq 0\}$.

The collection of all delta sequences is usually denoted as Δ.

Preposition 1. Let $\{\delta_n\} \in \Delta$, then we have

$$FR\delta_n(y) = \int_{-\infty}^{\infty} \delta_n(x)\cos(xy)dx \to 1 \; as \; n \to \infty \quad 6$$

and

$$FI\delta_n(y) = \int_{-\infty}^{\infty} \delta_n(x)\sin(xy)dx \to 0 \; as \; n \to \infty. \quad 7$$

Let $L^1(R), L^1(R) = L^1$, be the space of complex valued Lebesgue integrable functions. From Preposition 1 we establish this theorem :

Theorem 2. Let $f \in L^1$ then we have $f_H \; (f * \delta_n)(y) \to f_H f(y) \; as \; n \to \infty$.

Proof Let $f \in L^1, \{\delta_n\} \in \Delta$, then using of (11) implies

$$f_H(f*\delta_n)(y) = \int_{-\infty}^{\infty} FI(f*\delta_n)(x)\cos\, xy + FR(f*\delta_n)(x)\sin\, xy))dx \quad 8$$

Since $(f*\delta_n)(\zeta) = \int_{-\infty}^{\infty} f(t)\delta_n(\zeta - t)dt \to f(\zeta) \; as \; n \to \infty$ we see that

$$FI(f*\delta_n)(x) = \int_{-\infty}^{\infty} (f*\delta_n)(\zeta)\sin(x\zeta)d\zeta$$

$$= \int_{-\infty}^{\infty} f(t)\,(\zeta - t)\sin(x\zeta)d\zeta dt$$

$$\to \int_{-\infty}^{\infty} f(t)\,\sin(xt)dt.$$

Similarly

$$FR(f*\delta_n)(x) \to FR(f)(x) \; as \; n \to \infty.$$

Therefore, invoking above equations in (26) we get $f_H(f*\delta_n)(y) \to f_H f(y) \; as \; n \to \infty$.

Hence the theorem is completely proved.

By β_{L^1} we denote the space of integrable Boehmians, then β_{L^1} is a convolution algebra when multiplication by scalar, addition and convolution are defined as [9]

$$k\left[\frac{f_n}{\delta_n}\right] = \left[\frac{kf_n}{\delta_n}\right], \left[\frac{f_n}{\delta_n}\right] + \left[\frac{g_n}{\gamma_n}\right] = \left[\frac{f_n*\gamma_n + g_n*\delta_n}{\delta_n*\gamma_n}\right]$$

and

$$\left[\frac{f_n}{\delta_n}\right] * \left[\frac{g_n}{\gamma_n}\right] = \left[\frac{f_n*g_n}{\delta_n*\gamma_n}\right]$$

Each function $f \in L^1$ is identified with the Boehmian $\left[\frac{f*\delta_n}{\delta_n}\right]$. Since $\left[\frac{\delta_n}{\delta_n}\right]$ corresponds to Dirac delta distribution δ, the kth-derivative of each $\rho \in \beta_{L^1}$ is defined as

$$D^k\rho = \rho * D^k\delta.$$

Following theorem has importance in the sense of analysis.

Theorem 3. Let $\left[\frac{f_n}{\delta_n}\right] \in \beta_{L'}$ then the sequence

$$f_H(f_n)(y) = \int_{-\infty}^{\infty} (FIf_n(x)\cos(xy) + FRf_n(x)\sin(xy))dx$$

converges uniformly on each compact subset K of R.

Proof. By aid of Theorem 2 and the concept of quotient of sequences we have

$$f_H(f_n)(y) = f_H\left(f_n * \frac{\delta_k}{\delta_k}\right)(y)$$

$$= f_H\left(\frac{f_n * \delta_k}{\delta_k}\right)(y)$$

$$= f_H\left(f_n * \frac{\delta_k}{\delta_k}\right)(y)$$

$$\to f_H\frac{f_k}{\delta_k}(y) \; as \; n \to \infty.$$

where convergence ranges over compact subsets of R.

The theorem is completely proved.

Let $\left[\frac{f_n}{\delta_k}\right] \in \beta_{L'}$ then by virtue of Theorem 3 we define the Fourier-Hilbert transform of the Boehmian $\left[\frac{f_n}{\delta_k}\right] \in \beta_{L'}$ as

$$\widetilde{f_H}\left[\frac{f_k}{\delta_k}\right] = \lim_{n\to\infty} f_n\,. \quad 9$$

on compact subsets of R.

Next objective is to establish that our definition is well-defined. For, let $\left[\frac{f_n}{\delta_n}\right] = \left[\frac{g_n}{\gamma_n}\right] \; in \; \beta_{L'}$ then

$$f_n * \gamma_m = g_n * \delta_n, \text{ for every } m, n \in N.$$

Hence, applying the Fourier-Hilbert transform to both sides of above equation and using concept of qoutients of sequences imply

$$f_H(f_n * \gamma_m) = f_H(g_n * \delta_n) = f_H(g_n * \delta_m)$$

In particular, for $n = m$, and considering Theorem 3 we get

$$lim_{n\to\infty} f_H f_n = lim_{n\to\infty} f_H g_n.$$

Hence,

$$\widetilde{f_H}\left[\frac{f_n}{\delta_n}\right] = \widetilde{f_H}\left[\frac{g_n}{\gamma_n}\right].$$

$\widetilde{f_H}$ is therefore well-defined.

Theorem 4. The transform $\widetilde{f_H}$ is linear.

Proof Let $\rho_1 = \left[\frac{f_n}{\delta_n}\right]$ and $\rho_2 = \left[\frac{g_n}{\gamma_n}\right]$ be arbitrary in $\beta_{L'}$ and $\alpha \in \mathbb{C}$, then

$$\rho_1 + \rho_2 = \left[\frac{f_n * \gamma_n + g_n * \delta_n}{\delta_n * \gamma_n}\right].$$

Hence, we get

$$\widetilde{f_H}(\rho_1 + \rho_2) = lim_{n\to\infty}\big(f_H(f_n * \gamma_n) + f_H(g_n * \delta_n)\big).$$

By Theorem 2 we get

$$\widetilde{f_H}(\rho_1 + \rho_2) = lim_{n\to\infty} f_H f_n + lim_{n\to\infty} f_H g_n$$

Hence

$$\widetilde{f_H}(\rho_1 + \rho_2) = \widetilde{f_H}\rho_1 + \widetilde{f_H}\rho_2$$

Further, if α is a complex number then, indeed,

$$\widetilde{f_H}(\alpha\rho_1) = \widetilde{f_H}\left[\frac{\alpha f_n}{\delta_n}\right] = \alpha lim_{n\to\infty} f_H f_n = \alpha \widetilde{f_H}\rho_1.$$

Hence the theorem is proved.

Theorem 5. Let $\rho \in \beta_{L'}$ and $\{\varepsilon_k\} \in \Delta$, then

$$\widetilde{f_H}(\rho * \varepsilon_n) = \widetilde{f_H}\rho = \widetilde{f_H}(\varepsilon_n * \rho)$$

Proof Let $\rho = \left[\frac{f_n}{\delta_n}\right] \in \beta_{L'}$, then we have

$$\widetilde{f_H}(\rho * \varepsilon_n) = \widetilde{f_H}\left[\frac{f_n * \varepsilon_n}{\delta_n}\right] = lim_{n\to\infty} f_H(f_n * \varepsilon_n)$$

Hence, $\widetilde{f_H}(\rho * \varepsilon_n) = lim_{n\to\infty} f_H f_n = \widetilde{f_H}\rho$

Similarly we proceed for $\widetilde{f_H}\rho = \widetilde{f_H}(\varepsilon_n * \rho)$.

This completes the theorem.

Following theorem is obvious.

Theorem 6. If $\widetilde{f_H}\ \rho_1 = 0$, then $\rho_1 = 0$.

Theorem 7. The Fourier-Hilbert transform $\widetilde{f_H}$ is continuous with respect to the δ −convergence.

Proof Let $\rho_n \overset{\delta}{\to} \rho$ in $\beta_{L'}$ as $n \to \infty$, then we show that $\widetilde{f_H}\rho_n \overset{\delta}{\to} \widetilde{f_H}\rho$ as $n \to \infty$. Using [15,Theorem 2.6] we find $f_{n,k}, f_k \in L'$, $\{\delta_k\} \in \Delta$ such that $\left[\frac{f_{n,k}}{\delta_k}\right] = \rho_n$, $\left[\frac{f_k}{\delta_k}\right] = \rho$ and $f_{n,k} \to f_k$ as $n \to \infty, k \in N$.

Applying the Fourier-Hilbert transform for both sides implies $f_H\ f_{n,k} \to f_H\ f_k$ in the space of continuous functions. Therefore, considering limits we get

$$\widetilde{f_H}\left[\frac{f_{n,k}}{\delta_k}\right] \to \widetilde{f_H}\left[\frac{f_k}{\delta_k}\right].$$

This completes the proof of the theorem.

Theorem 8. The Fourier-Hilbert transform $\widetilde{f_H}$ is continuous with respect to the Δ −convergence.

Proof. Let $\rho_n \overset{\Delta}{\to} \rho$ as $n \to \infty$ in $\beta_{L'}$ then there is $\{f_n\} \in L'$ and $\{\delta_k\} \in \Delta$ such that

$(\rho_n - \rho) * \delta_n = \left[\frac{f_n * \delta_k}{\delta_k}\right]$ and $f_n \to 0\ as\ n \to \infty$

Thus by aid of Theorem 3 and the hypothesis of the theorem we have

$$\widetilde{f_H}((\rho_n - \rho) * \delta_n) = \widetilde{f_H}\left[\frac{f_n * \delta_k}{\delta_k}\right] \to f_H\,(f_n * \delta_k) as\ n \to \infty \to f_H f_n\ as\ n \to \infty \to 0\ as\ n \to \infty\ .$$

Therefore $\widetilde{f_H}(\rho_n - \rho) \to 0$ as $n \to \infty$. Thus $\widetilde{f_H}\rho_n \overset{\Delta}{\to} \widetilde{f_H}\rho$ as $n \to \infty$.

This completes the proof.

Lemma 9. Let $\left[\frac{f_n}{\delta_n}\right] \in \beta_{L'}$and δ is the delta distribution, then we have

$$\widetilde{f_H}\left(\left[\frac{f_n}{\delta_n}\right] * \delta\right) = \widetilde{f_H}\left[\frac{f_n}{\delta_n}\right].$$

Proof Let $\rho = \left[\frac{f_n}{\delta_n}\right] \in \beta_{L^1}$, then we have

$$\widetilde{f_H}\left(\left[\frac{f_n}{\delta_n}\right] * \delta\right) = \widetilde{f_H}\left[\frac{f_n * \delta}{\delta_n}\right] = lim_{n\to\infty}\ f_H(f_n * \delta) = lim_{n\to\infty}\ f_H f_n.$$

Hence

$$\widetilde{f_H}\left(\left[\frac{f_n}{\delta_n}\right] * \delta\right) = \widetilde{f_H}\left[\frac{f_n}{\delta_n}\right].$$

Theorem 10. The Fourier-Hilbert transform $F\sim_H$is one-to-one.

Proof Let $\widetilde{f_H}\left[\frac{f_n}{\delta_n}\right] = \widetilde{f_H}\left[\frac{g_n}{\gamma_n}\right]$ then we get $lim_{n\to\infty}\ f_H f_n = lim_{n\to\infty}\ f_H g_n$. Hence $f_H(lim_{n\to\infty}\ f_n) = f_H(lim_{n\to\infty}\ g_n)$. That is $f_H f = f_H g$. The fact that F_H is one-to-one implies $f = g$.

Hence the theorem is completely proved.

III. Fourier-Hilbert Transform of Distributions

Denote by $C(R)$ the space of smooth functions and $C'(R)$ the strong dual of C of distributions of compact support over R.

Then, we have the following convolution theorem for f_H.

Theorem 11. (Convolution Theorem) Let f and $g \in C$ then we have

$$f_H(f * g)(y) = \int_0^\infty (k_1(x)cos(yx) + k_2(x)sin(yx))dx \qquad 10$$

where

$$k_1(x) = FRf(x)FIg(x) + FIf(x)FRg(x)$$
$$and$$
$$k_2(x) = FRf(x)FRg(x) - FIf(x)FIg(x).$$

Proof To prove this theorem it is sufficient to establish that $k_1(x) = FI(f * g)(x)\ and\ k_2(x) = FR(f * g)(x)$. We have

$$FI(f * g)(x) = \int_{-\infty}^{\infty} (\int_{-\infty}^{\infty} (f(\gamma)g(y - \gamma)dr)cos(xy) + sin(xy))dy = \int_{-\infty}^{\infty} f(\gamma) \int_{-\infty}^{\infty} g(y - \gamma)(cos(xy) + sin(xy))dydr$$

By change of variables and parity Fubiniz Theorem implies

$$FI(f * g)(x) = \int_{-\infty}^{\infty} f(\gamma) \int_{-\infty}^{\infty} g(z)\big(cos(x(z + \gamma)) + sin(x(z + \gamma))\big)dzdr.$$

Taking into account the formulas $cos(x(z + \gamma)) = cos(xz)cos(x\gamma) - sin(xz)sin(x\gamma)$ and $sin(x(z + \gamma)) = sin(xz)cos(x\gamma) + cos(xz)sin(x\gamma)$, Equation (17) follows from simple computation.

Hence the theorem is completely proved.

The fact that $cos(xy), sin(xy) \in C$ gives $FIf, FRf \in C'$. Hence, we have the following statement.

Definition 12. Let $f \in C'$ then we define the distributional Fourier-Hilbert transform of f as

$$\widehat{f_H}f(y) = \langle FIf(x), cos(xy)\rangle + \langle FRf(x), sin(xy)\rangle.$$

The extended transform $\widehat{f_H}f$ is clearly well-defined for each $f \in C'$.

Theorem 13. The distributional Fourier-Hilbert transform $\widehat{f_H}f$ is linear.

Proof. Let $f, g \in C'$ then their components $FRf, FIf, FRg, FIg \in C'$. Hence,

$$\widehat{f_H}(f + g)(y) = \langle FI(f + g)(x), cos(xy)\rangle + \langle FR(f + g)(x), sin(xy)\rangle.$$

By factoring and rearranging components we get that

$$\widehat{f_H}(f + g)(y) = \widehat{f_H}f(y) + \widehat{f_H}g(y).$$

Further,

$$\widehat{f_H}(kf)(y) = \langle kFIf(x), cos(xy)\rangle + \langle kFIf(x), sin(xy)\rangle +$$

Hence

$$\widehat{f_H}(kf)(y) = k\widehat{f_H}f(y).$$

This completes the proof of the theorem.

Theorem 14. Let $f \in C'$ then the mapping $\widehat{f_H}f$ is continuous .

Proof. Let $\{f_n\}, f \in C', n \in N$ and $f_n \to f$ as $n \to \infty$. Then, we have

$$\widehat{f_H}f_n(y) = \langle FIf_n(x), cos(xy)\rangle + \langle FRf_n(x), sin(xy)\rangle \to \langle FIf(x), cos(xy)\rangle + \langle FRf(x), sin(xy)\rangle = \widehat{f_H}f(y)\ as\ n \to \infty.$$

Hence the theorem is completely proved.

Theorem 15 The mapping $\widehat{f_H}f$ is one-to-one.

Proof. Let $f, g \in C'$ and that $\widehat{f_H}f = \widehat{f_H}g$, then on account of (11) we get

$$\langle FIf(x), cos\ xy\rangle + \langle FRf(x), sin\ xy\rangle = \langle FIg(x), cos\ xy\rangle + \langle FRg(x), sin\ xy\rangle$$

Basic properties of inner product implies

$$\langle FIf(x) - FIg(x), cos(xy)\rangle + \langle FRf(x) - FRg(x), sin(xy)\rangle = 0.$$

Hence $FIf(x) = FIg(x)\ and\ FRf(x) = FRg(x)$.

Therefore

$$f_Hf(x) = FIf(x) + FRf(x) = FIg(x) + FRg(x) = Ag(x)$$

for all x.

This completes the proof of the theorem.

Theorem 16. Let $f \in C'$, then f is an analytic mapping and

$$D_y^k\widehat{f_H}f(y) = \langle FIf(x), D_y^k cos(xy)\rangle + \langle FRf(x), D_y^k sin(xy)\rangle.$$

Proof is straightforward. Detailed proof is therefore omitted.

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