International Scientific Journal
Journal of Mathematics
http://mathematics.scientific-journal.com

Solution of Lane-Emden Type Equations Using Polynomial-Sinc Collocation Method

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Abstract

In this paper we introduce a series solution for Lane-Emden type equations. This solution based on Lagrange polynomials that can easily deal with singularity problems. The proposed approximation is based on non-equidistant interpolation points generated by conformal maps. Our method provides the solution by an exponential convergent series. This exponential convergence property arises from the use of Sinc points as interpolation points in the Lagrange polynomials. We examine the technique for different types of Emden' equations and compare the solution with exact solution and Taylor approximation.

1 Introduction

The Lane-Emden (LE) equation is one of the basic equations in the theory of stellar structure and has been the focus of several studies [1, 2, 3]. In general, LE type equations are nonlinear ODEs that can be formulated as:

$$y'' + \frac{\kappa}{x} y' + g(y) = R(x), \;\; x > 0 \tag{1}$$

with the initial conditions,

$$y(0) = a, \, y'(0) = b.$$

In general, $g(y)$ is a nonlinear function of y, $R(x)$ is a function of x only and, κ is a positive integer. Many problems in mathematical physics and astrophysics are related to this equation. Some of these applications are homogeneous, $R(x) = 0$, and others are inhomogeneous. For every physical application a suitable choice of the generic function $g(y)$ is made.

To motivate the astrophysical background of LE, let us consider a spherical cloud of gas and denote its hydrostatic pressure at a distance r from the center by P. Let $M(r)$ be the mass of a star, and ρ its density. Then Poisson's equation and the condition for hydrostatic equilibrium can be stated as [See [1]]:

$$\frac{dP}{dr} = -\rho \frac{GM(r)}{r^2}, \tag{2}$$

$$\frac{dM(r)}{dr} = 4\pi\rho r^2, \tag{3}$$

where G is the gravitational constant. Combining (2) and (3), an equivalent form of the Poisson equation can be written as:

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G\rho. \tag{4}$$

Assuming that P is dependent on the density ρ and independent of the temperature, the polytropic pressure relation can be stated as

$$P = k\rho^{1+\frac{1}{m}}, \tag{5}$$

where k is a constant and m is the poly tropic index related to the ratio of specific heats of the gas comprising the star.

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Let λ represents the central density of the star and y is a dimensionless quantity that are related to ρ by the following relation:

$$\rho = \lambda y^m. \tag{6}$$

We are able to standardize the polytropic pressure equation by inserting (5) and (6) in (2) to find

$$\left[\frac{k(m+1)}{4\pi G}\lambda^{\frac{1}{m}-1}\right]\frac{1}{r^2}\frac{d}{dr}(r^2\frac{dy}{dr}) = -y^m. \tag{7}$$

Finally, let us define a dimensionless variable x as:

$$r = \sqrt{\frac{k(m+1)}{4\pi G}\lambda^{\frac{1}{m}-1}}x. \tag{8}$$

Then (7) will be,

$$y'' + \frac{2}{x}y' + y^m = 0,\ x > 0, \tag{9}$$

with the conditions,

$$y(0) = 1,\ y'(0) = 0,$$

which is the standard form of the LE equation introduced in (1) with parameter m an integer taken from the interval $[0, 5]$.

One of the main problems with the LE equation is the singularity at $x = 0$ which is a singularity at the boundary as well as of the equation. This singularity was a challenge for many scholars to numerically represent the solution of the LE equation. The problem of a reliable representation of the solution was actually discussed by both Lane and Emden [4, 5] in their earlier examinations. Since its establishment the importance of the equation (9), number of applications increased during the years. Even today there is an increasing need to find a reliable and accurate way to approximate its solutions.

The standard LE equation has analytic solutions for $m = 0$, 1, and 5. For the other values of m, to our best knowledge, analytic solutions are missing and thus the equation must be integrated numerically to get at least an approximate solution.

The LE equation was examined with respect to the numerical stability by many authors, for example see [6, 7, 21] and references therein. On the other hand analytic examinations of LE equations as well were performed by different groups [2, 8].

Most of the techniques used to solve the LE equations and its types are based on either series solution, Adomain decomposition or perturbation techniques [8, 9, 10]. Although the series solutions can be obtained by monotonous computations but they must be modified to adapt the singularity at $x = 0$. This singularity is not always easy to handle especially on finite intervals. For example, when using series approximations on a finite interval the Runge phenomenon may arise to give an inaccurate approximation near the end points of the intervals [11]. To overcome the Runge phenomenon, the use of Chebyshev polynomial approximation has become an accurate way to solve many problems in applications. The results of a Chebychev approximation are usually very accurate, except if singularities are present at end-points of the interval of approximation [6].

Another approach to tackle the problem of singularity is the Adomain decomposition method. This method is currently used by some authors [8, 10] to compute the series solution for the LE but also applicable to other models than LE. The Adomain decomposition method accurately computes the series solution in a rapidly convergent way.

The core question again is: how can the series approximation method be modified to address the problem of singularity. The offered answers vary from equation to equation and sensitively depends on the type of singularity [9, 10].

In this paper, we aim to introduce a Lagrange polynomial approximation at some non-equidistant points, so called Sinc points. We also aim to define a collocation scheme to establish Polynomial-Sinc, in short Poly-Sinc, approximation. Finally we use this scheme to solve well-known LE type equations classified by (1) for various $g(y)$ and $R(x)$.

This paper is organized as follows: In section 2, the Poly-Sinc interpolation formula is described. In section 3, we introduce the Poly-Sinc algorithm based on the approximation proposed in section 2. In section 4, we apply the Poly-Sinc algorithm to solve the standard LE equation. In section 5, solutions of classes of homogeneous and inhomogeneous differential equations of LE type are discussed. Concluding remarks are given in section 6.

2 Poly-Sinc Interpolation

Different sets of points $\{x_k, f(x_k)\}_{k=0}^{n}$ are used as interpolation points in Lagrange polynomial approximation [12]. The most famous set of points are the equidistant points. It is well known that these points deliver bad results [13]. To improve the accuracy of Lagrange approximation other sets of points are used, like Chebychev points and modified Chebychev points [12]. Recently it was shown that it is more effective to use Sinc points as interpolation points [14, 25]. This sequence of points is created using a conformal map that redistribute the infinite equidistant points of the real line on a finite interval locating most of these points near the end-points of the finite interval. It is also proved that using Sinc-points as interpolation points will deliver a high accurate approximation and allows an accuracy similar to the classical Sinc approximation [14].

To define these interpolation points let $\mathbb{Z}$ denote the set of all integers. Let $\mathbb{R}$ be the real line, and $\mathbb{C}$ denote the complex plane. Let h denote a positive parameter and let $k \in \mathbb{Z}, \;\; x \in \mathbb{C}$. In addition, let d denote a positive number, and let ϕ denote the conformal map of a simply connected region $\mathcal{D} \subset \mathbb{C}$ onto the strip

$$\mathcal{D}_d = \{z \in \mathbb{C} : |Im(z)| < d\}.$$

Let $\Gamma = \phi^{-1}(\mathbb{R})$ be an arc and let $a = \phi^{-1}(-\infty)$ and $b = \phi^{-1}(\infty)$ denote the end points of Γ. Then we define the set of Sinc points by $x_k = \phi^{-1}(kh)$, and set $\rho = e^{\phi(x)}$.

Finally, let $\alpha \in (0, 1]$ and $\beta \in$(0,1] denote fixed positive numbers. Without loss of generality, let us restrict d introduced above to the interval $(0, \pi)$. Let $\mathcal{L}_{\alpha,\beta}(\mathcal{D})$ denote the family of all functions that are analytic in $\mathcal{D}$, such that for all $z \in \mathcal{D}$, we have

$$|y(z)| \leq c_1 \frac{|\rho(z)|^{\alpha}}{[1 + |\rho(z)|]^{\alpha+\beta}}.$$

The space of functions $M_{\alpha,\beta}(\mathcal{D})$ denotes the set of all functions q defined on $\mathcal{D}$ that have finite limits $q(a) = \lim_{z\to a} h(z)$ and $q(b) = \lim_{z\to b} q(z)$, where the limits are taken from within $\mathcal{D}$, and such that $y \in \mathcal{L}_{\alpha,\beta}(\mathcal{D})$, where,

$$y = q - \frac{q(a) + \rho q(b)}{1 + \rho}.$$

Now we are in position to define a set of Sinc data of the form $\{x_k,\; y\,(x_k)\}_{k=-M}^{N}$ where the x_k are Sinc points and $\{y\,(x_k)\}_{k=-M}^{N}$ are the function values. Let x_k be the Sinc points defined on the interval $[a, b]$. These points are generated using the conformal map $\phi(x) = \ln((x - a)/(b - x))$ and so the Sinc points can be given by:

$$x_k = \phi^{-1}(kh) = \frac{a + be^{kh}}{1 + e^{kh}},\; k = -M, ..., N. \qquad (10)$$

Next we define a family of polynomial-like approximation that interpolate given Sinc data of the form $\{x_k, y(x_k)\}_{k=-M}^{N}$ where the x_k are Sinc points defined in (10). This novel family of Lagrange polynomials was recently derived in [18]. The approximation is accurate, provided that the function y with $y_k = y(x_k)$ belongs to the space of analytic functions $\mathcal{L}_{\alpha,\beta}(\mathcal{D})$.

Generally Lagrange polynomial approximation over the interval $[a, b]$ is defined in the following way. Given a set of $n = M+N+1$ Sinc points $\{x_k\}_{k=-M}^{N}$on the interval $[a, b]$ as defined in (10) and function values, $\{y(x_k)\}_{k=-M}^{N}$. At these points $\{x_k, y(x_k)\}_{k=-M}^{N}$, there exists a unique polynomial $p(x)$ of degree at most $n - 1$ satisfying the interpolation condition,

$$p(x_k) = y_k,\; k = -M,\; ..., N.$$

In this case $p(x)$ can be expressed as:

$$p(x) = \sum_{k=-M}^{N} b_k(x) y_k, \qquad (11)$$

with,

$$b_k(x) = \frac{v(x)}{(x - x_k)v'(x_k)}, \qquad (12)$$

where,

$$v(x) = \prod_{j=-M}^{N} (x - x_j)\,.$$

This approximation, like regular Sinc approximation, yields an exceptional accuracy in approximating the function that is known at Sinc points [15]. Unlike Sinc approximation, it gives an exponential convergence rate when differentiating the interpolation formula given in (11), [14].

For the sake of simplicity of our results, we shall assume here that $M = N$.

Theorem 1. *Let $h = \frac{\pi}{\sqrt{N}}$, and let $\{x_k\}_{k=-N}^{N}$ denote the Sinc points as defined in (10). Let y be in $M_{\alpha,\beta}(\mathcal{D})$, and let $p(x)$ be defined as in (11). Then there exist two constants $A > 0$ and $B > 0$, independent of N, such that*

$$|y(x) - p(x)| \leq A \frac{\left(\sqrt{N}\right)}{B^{2N}} \ \exp\left(\frac{-\pi^2 N^{\frac{1}{2}}}{2}\right), \quad (13)$$

Proof 1. *For the proof of (13), see [14].*

The space $M_{\alpha,\beta}(\mathcal{D})$ is connected with a Hardy space $H^p(\mathcal{U})$, where $\mathcal{U}$ is the unit disk. In $H^p(\mathcal{U})$ the best approximation rate of the form $O\left(e^{-c\sqrt{N}}\right)$ has been obtained [16]. The constant $c = \frac{\pi}{2}$ in our own estimate is not large as this optimal upper bound of the error discussed in [16] as $c = \pi$, but it is still an excellent upper bound comparing with the upper bounds in Sinc approximations. The optimal distribution of the sequence of points to be chosen on the interval $[a, b]$ is still an open problem till now [13, 16], but one can see that the approximation we introduced here give an exceptional upper bound of error. In fact getting an exponential decaying rate of convergence is not only the privilege of this approximation, but dealing effectively without any modification with the singularity at the end points.

3 Poly-Sinc Algorithm

In this section we set up a collocation method based on Lagrange interpolation at Sinc points. To do this, let us consider the nonlinear differential equations of order 2 defined on the interval $(a,\, b)$ as:

$$L(u) \equiv u'' + l_1(x,u)u' + l_2(x,u) = f(x),\ a \leq x \leq b. \quad (14)$$

With the boundary conditions:

$$u(a) = u_0 \text{ and } u(b) = u_1, \quad (15)$$

where the functions $l_1(x,u)$, and $l_2(x,u)$ are functions of x and u which are nonlinear (or linear), and singular (or non-singular). The existence and uniqueness of the solution of (14-15) is discussed by Agarwal and Akrivis in [17].

The idea of the Poly-Sinc algorithm is to replace $u(x)$ in equation (14) and (15) by the Lagrange polynomial defined in (11). This will reduce the problem to a system of $2N + 1$ algebraic equations.

To set up the Poly-Sinc algorithm, we will present it in details using the following Mathematica lines. To do this we start the algorithm by defining the conformal mapping and Sinc points on the interval $[a,\, b]$.

Defining the conformal map $\phi(x)$ as,

$\boldsymbol{\phi}$**[x_]:=Log** $\left[\frac{x-a}{b-x}\right]$

Where its inverse conformal map $\psi(x) = \phi^{-1}(x)$ can be defined as,

$\boldsymbol{\psi}$**[x_]:=** $\frac{be^{x}-a}{1+e^{x}}$

The Sinc points are defined by calculating $\psi(x)$ at $x = kh$

Nu = val1; Ml = val2;

where, val1 and val2 are positive integers.

sp[k_]:= $\psi[kh]/.h \to \frac{\pi}{\sqrt{\mathbf{Nu}}}$**;**

We then define the Poly-Sinc interpolation formula (11). First we need to define the function $v(x)$ and its derivative at Sinc points, $v'(x)$.

For $v(x)$,

v**[x_]:=Product[(** x **− sp[** l **]), {** l **, −Ml, Nu}]**

Differentiate $v(x)$ with respect to x to get the function $dv(x)$

$\mathbf{dv[x\_]:=\partial_x v[x]}$

Calculating $dv(x)$ at Sinc points as,

$\mathbf{dvs[i\_]:=dv[x]/.x \to sp[i]}$

Now defining the basis functions $b_k(x)$ as,

$\boldsymbol{b}\mathbf{[x\_, k\_, N\_, M\_]:=}\boldsymbol{v[x]/((x - }\mathbf{sp}\boldsymbol{[k])}\mathbf{dvs}\boldsymbol{[k])}$

The Lagrange interpolation formula at Sinc data, (11), can be define by the following line,

$\boldsymbol{p}\mathbf{[x\_, N\_, M\_]:=}\sum_{k=-M}^{N} \boldsymbol{b[x, k, N, M]u[k]}$

The first step to solve (14–15) is to replace $u(x)$ in (14) by the polynomial $p(x)$ defined in (11),

$$\mathbf{stepColl1} = \boldsymbol{N}[\boldsymbol{\partial_{x,x}p}[\boldsymbol{x}, \mathbf{Nu}, \mathbf{Ml}] + \boldsymbol{l_1}[\boldsymbol{x}, \boldsymbol{p}[\boldsymbol{x}, \mathbf{Nu}, \mathbf{Ml}]]\boldsymbol{\partial_x p}[\boldsymbol{x}, \mathbf{Nu}, \mathbf{Ml}] + \boldsymbol{l_2}[\boldsymbol{x}, \boldsymbol{p}[\boldsymbol{x}, \mathbf{Nu}, \mathbf{Ml}]] == \boldsymbol{f}[\boldsymbol{x}]];$$

The second step is to evaluate the result at the Sinc points $x_j = \psi(jh), j = -M+1, ..., N-1$. This delivers a nonlinear system of $2N-1$ equations.

$$\mathbf{stepColl2} = \mathbf{Map}[(\mathbf{stepColl1}/.\boldsymbol{x} \to \#)\&, \mathbf{Table}[\mathbf{sp}[\boldsymbol{k}], \{\boldsymbol{k}, -(\mathbf{Ml}-\mathbf{1}), \mathbf{Nu}-\mathbf{1}\}]];$$

Now to the boundary conditions in (15),

$$\mathbf{B1} = (\boldsymbol{p}[\boldsymbol{a}, \mathbf{Nu}, \mathbf{Ml}]//\boldsymbol{N}) == \boldsymbol{u_0};$$

$$\mathbf{B2} = (\boldsymbol{p}[\boldsymbol{b}, \mathbf{Nu}, \mathbf{Ml}]//\boldsymbol{N}) == \boldsymbol{u_1};$$

Combining the $(2N-1)$ equation from stepCollocation2 and the collocated boundary conditions, boundary1 and boundary2, to get a system of $(2N+1)$ of algebraic equations,

$$\mathbf{AppendTo}[\mathbf{stepColl2}, \{\mathbf{B1}, \mathbf{B2}\}];$$

$$\mathbf{stepColl3} = \mathbf{Flatten}[\mathbf{stepColl2}];$$

Now the differential equation transformed to a system of $(2N+1)$ algebraic equations in $(2N+1)$ unknowns, $u[k]$, $k = -N, ..., N$. Creating the $2N+1$ algebraic system, the discretized form of $p'(x)$ at Sinc points, $x_j = \psi(jh)$ is needed. To get $p'(x_j)$, we differentiate (11) with respect to x and then evaluate the result at the Sinc points x_j to get.

$$f'(x_j) \approx p'(x_j) = \sum_{k=-N}^{N} a_{j,k}\, f(x_k)\,, \tag{16}$$

where $a_{j,k}$, $j, k = -N, ..., N$ defines a $n \times n, n = 2N+1$ matrix $\mathbf{A}$ defined as,

$$\mathbf{A} = \begin{cases} \frac{v'(x_j)}{(x_j - x_k)v'(x_k)} & k \neq j. \\ \sum_{l=-M, l\neq j}^{N} \frac{1}{x_j - x_l} & k = j. \end{cases} \tag{17}$$

Also the discretized form of $p''(x)$, $p''(x_j)$ with $x_j = \psi(jh)$, is required. To get $p''(x_j)$, we differentiate (11) twice with respect to x and then evaluate the result at the Sinc points x_j to get,

$$p''(x_j) = \sum_{k=-N}^{N} c_{j,k} u_k, \tag{18}$$

where a matrix $C = c_{j,k}$ defined as,

$$\mathbf{C} = \begin{cases} \frac{-2v'(x_j)}{(x_j - x_k)^2 v'(x_k)} + \frac{v''(x_j)}{(x_j - x_k)v'(x_k)} & \text{if } k \neq j \\ \sum_{n=-N}^{N} \sum_{\substack{l=-N \\ l,n\neq j}}^{N} \frac{1}{(x_j - x_l)(x_j - x_n)} & \text{if } k = j. \end{cases} \tag{19}$$

Now we are in a position to solve the system of algebraic equations. To solve this system we use Newton root finding method to evaluate the coefficients, $u[k]$. Finally inserting the coefficients values, $u[k]$, in the approximate polynomial $p[\mathrm{x}, \mathrm{Nu}, \mathrm{Ml}]$ to get the Poly-Sinc approximate solution for the equation (14–15).

For practical purposes, we will use two forms of errors:

- **Absolute Error** The Absolute error is given by:

$$E = |u(x) - u_{ap}(x)| \tag{20}$$

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- **Norm Error** the norm error is given by:

$$\epsilon = \|u(x) - u_{ap}(x)\| = [\int_a^b (u(x) - u_{ap}(x))^2 \, dx]^{\frac{1}{2}}, \tag{21}$$

where $u(x)$ is the exact (or analytic) solution and $u_{ap}(x)$ is the approximation obtained from the Poly-Sinc algorithm.

4 Solution of Standard LE-Equation

In this section we examine the Poly-Sinc collocation algorithm to solve the standard LE equation. This equation is used to model the thermal behavior of a spherical cloud of gas acting under the mutual attraction of its molecules and subject to the classical laws of thermodynamics [1, 4]. The standard equation can be given by:

$$y'' + \frac{2}{x}y' + y^m = 0, x \in [0, x_1], \tag{22}$$

with the boundary conditions:

$$y(0) = 1,\ y'(0) = 0 \,\text{and},\ y(x_1) = y_1.$$

This equation allows analytic solutions for $m = 0, 1$ and, 5 given by:

$$y = 1 - \frac{x^2}{6},\ y = \frac{\text{Sin}(x)}{x}, \text{ and } y = \left(1 + \frac{x^2}{3}\right)^{\frac{-1}{2}}, \tag{23}$$

respectively. To our best knowledge these three solutions are the only analytic solutions known in literatures [4, 5]. For $m = 2, 3, 4$ there are many papers discussing different numerical techniques to get approximate solutions [6, 7, 8, 18, 19].

To check the validity of Poly-Sinc algorithm defined in section 3, let's start with the case when $m = 1$ and compare the approximate solution with the given exact solution.

- **For $m = 1$:**
The target here is to apply the Poly-Sinc algorithm introduced in section 3 to get an approximate solution for,

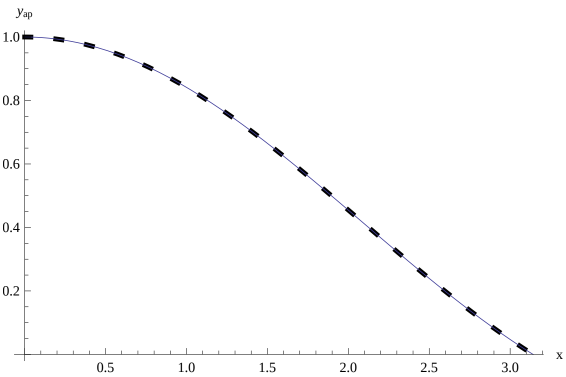

Figure 1: Exact and approximate solution of standard LE, (24).

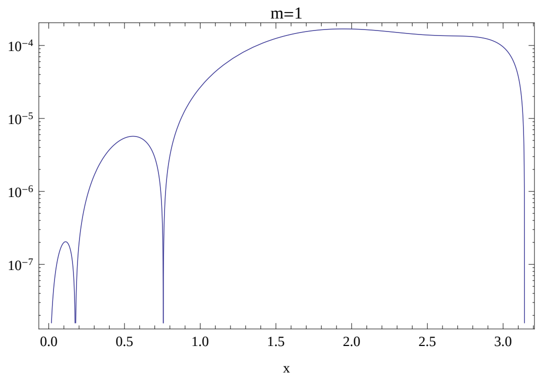

Figure 2: $|y - y_{ap}|$ for (24) with $N = 3$.

$$y'' + \frac{2}{x}y' + y = 0, x \in (0, \pi], \tag{24}$$
$$y(0) = 1,\ y'(0) = 0 \,\text{and},\ \ y(\pi) = 0.$$

Which has the exact solution,

$$y = \frac{\sin(x)}{x}$$

Figure 1 represents the exact solution, solid line, and approximate solution, dashed line.

Figure 2 displays the absolute error as a logarithmic plot using $N = 3$ in the Poly-Sinc calculations.

Calculating the norm error ϵ based on the norm using (21) results to 10^{-4} for this small number of collocation points, 7 Sinc points.

To illustrate the exponential convergence rate of the Poly-Sinc algorithm, we examined the error estimation form (13), $\gamma\sqrt{N}E^{-\mu\sqrt{N}}$, where γ and μ are constants. We need first to build a list of errors for different numbers of Sinc points as follow,

$$\mathbf{error = \{\};}$$

$$\mathbf{error1 = AppendTo[error, \{15, normerror\}];}$$

$$\{\{5, 0.00551426\}, \{7, 0.000190525\},$$
$$\{9, 4.72421573 \times 10^{-6}\}, \{11, 7.6947705067 \times 10^{-8}\},$$
$$\{13, 6.75211944171\times10^{-10}\}, \{15, 5.7521195\times10^{-11}\}\},$$

Now, defining the error function of the error, $\gamma\sqrt{N}E^{-\mu\sqrt{N}}$:

$$\mathbf{Expfunction} = \boldsymbol{\gamma\sqrt{x}e^{-\mu x^{0.5}}}$$

The point now is to find the values of the constants μ and γ that fit the error list created above,

$$\mathbf{parameters = FindFit[error1, Expfunction, \{\gamma, \mu\}, x, Method \rightarrow NMinimize]}$$

$$\{\gamma \rightarrow 233857., \mu \rightarrow 8.49929\}$$

So the error function is,

$$\mathbf{Fitfunction = Expfunction/.para}$$

$$233857.\sqrt{x}.e^{-8.49929x^{0.5}}$$

Figure 3 demonstrates that the error of the Poly-Sinc algorithm has an exponentially decaying rate.

The approximation of the LE equation for $m = 1$ demonstrates that the Poly-Sinc algorithm works properly. The verification of the $m = 1$ case puts us in a position to discuss the other cases for the parameter $m = 2, 3, 4$ that have no exact solutions. For these cases, $m = 2, 3, 4$, we apply the poly-Sinc

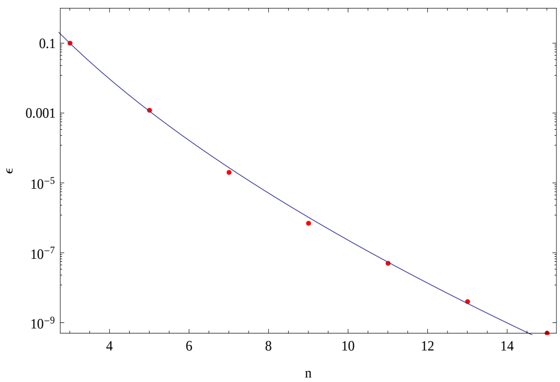

Figure 3: The error for different Sinc points $n = 3, 5, 7, 11, 13, 15$ for (24).

algorithm introduced in section 3. For the comparison purpose, we consider the series solution derived by Airey [20]. He derived series solution of equation (22) by applying a Taylor expansion about $x = 0$ for different values of $m \in [0, 5]$. The following solution for y was obtained using Airey's approach for general $m \in [0, 5]$:

$$\begin{aligned} y(x) &= 1 - \frac{x^2}{3!} + m\frac{x^4}{5!} + (5m - 8m^2)\frac{x^6}{3.7!} \\ &\quad + (70m - 183m^2 + 122m^3)\frac{x^8}{9.9!} \\ &\quad + (3150m - 1080m^2 + 1264m^3 \\ &\quad - 5032m^4)\frac{x^{10}}{45.11!} + ... \end{aligned} \tag{25}$$

Here, the expansion will carried out by us over a suitable range of $x \in [0, x_1]$ where $y(x_1) = y_1$.

- **For $m = 2$**

  Figure 4 shows the approximate solution using Poly-Sinc algorithm, dashed line, and the series solution defined in (25), solid line.

  Figure 5 represents the absolute error $|y - y_{ap}|$, where y is the series solution defined in (25) and y_{ap} is the solution using Poly-Sinc algorithm using 13 Sinc points.

  And calculating the norm error to get, $\epsilon = 7.39605 * 10^{-6}$.

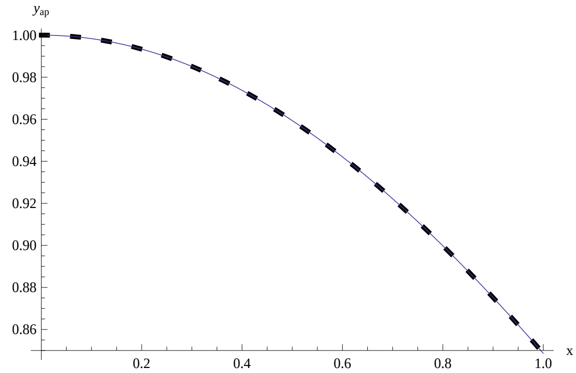

Figure 4: The analytic and approximate solution for LE with $m = 2$ and $N = 6$.

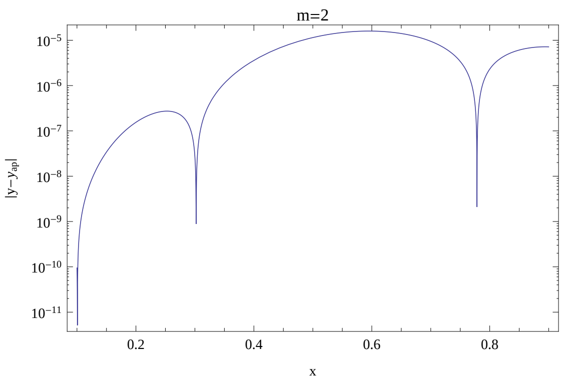

Figure 5: The absolute error in case $m = 2$ and $N = 6$.

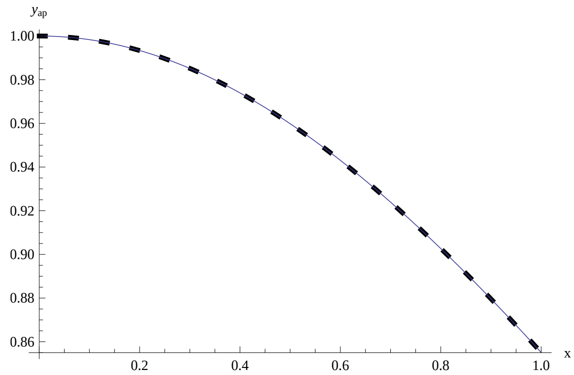

Figure 6: The analytic and approximate solution for LE with $m = 3$ and $N = 6$.

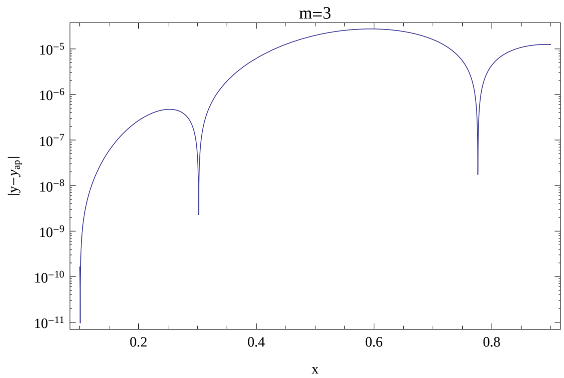

Figure 7: The absolute error in case $m = 3$ and $N = 6$.

- **For $m = 3$**

  Figure 6 shows the approximate solution using Poly-Sinc algorithm, dashed line, and the series solution defined in (25), solid line.

  Figure 7 represent the absolute error $|y - y_{ap}|$, where y is the series solution defined in (25) and y_{ap} is the solution using Poly-Sinc algorithm using 13 Sinc points.

  And calculating the norm error to get, $\epsilon = 1.27518 * 10^{-5}$.

- **For $m = 4$**

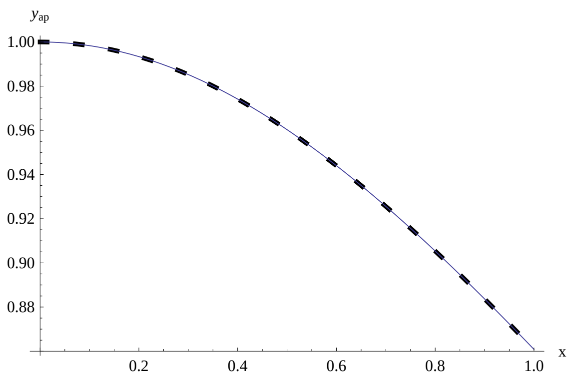

Figure 8: The analytic and approximate solution for LE with $m = 4$ and $N = 6$.

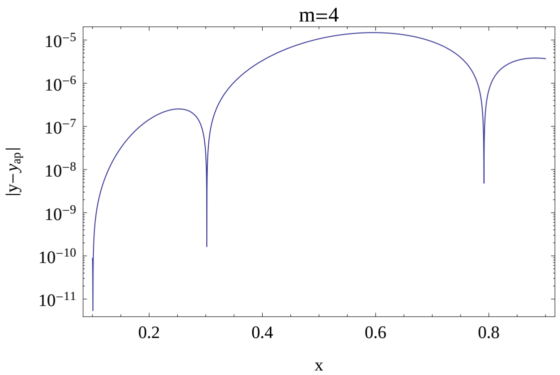

Figure 9: The absolute error in case $m = 4$ and $N = 6$.

Figure 8 shows the approximate solution using Poly-Sinc algorithm, dashed line, and the series solution defined in (25), solid line.

Figure 9 represent the absolute error $|y - y_{ap}|$, where y is the series solution defined in (25) and y_{ap} is the solution using Poly-Sinc algorithm using 13 Sinc points.

And calculating the norm error to get, $\epsilon = 6.6599 * 10^{-6}$.

Finally we collect all of the cases for $m \in [0, 5]$ with the norm error in each case in table 1. For all the calculations we apply Poly-Sinc algorithm with $n = 13$ Sinc points.

m	Equation	Interval	ϵ
0	$y'' + \frac{2}{x}y' + 1 = 0$	$[0, 1]$	10^{-7}
1	$y'' + \frac{2}{x}y' + y = 0$	$[0, \pi]$	10^{-10}
2	$y'' + \frac{2}{x}y' + y^2 = 0$	$[0, 1]$	10^{-6}
3	$y'' + \frac{2}{x}y' + y^3 = 0$	$[0, 1]$	10^{-5}
4	$y'' + \frac{2}{x}y' + y^4 = 0$	$[0, 1]$	10^{-6}
5	$y'' + \frac{2}{x}y' + y^5 = 0$	$[0, 1]$	10^{-12}

Table 1: The obtained error using (21) for standard LE for $m = 0, 1, 2, 3, 4$, and 5, using 13 Sinc points

Note:
The solution of the standard LE-equation showed two points of interest in using Poly-Sinc technique in solving the nonlinear singular boundary value problems. These two points are:

1. Using a small number of Sinc-points we can reach a very small error level. In comparison with other collocation techniques we can get the same or even better error using a much smaller number of collocation points. For example, in [21] a collocation technique based on Hermit polynomial was introduced to solve the standard LE equation for $m = 3$ and $m = 4$. With 31 collocation points an error of 10^{-4} has been obtained, while using 13 collocation points in Poly-Sinc algorithm we can get 10^{-5} and 10^{-6} respectively.

2. The numerical investigation of Poly-Sinc algorithm for solving standard LE-equation show that it has an exponentially decaying error property. This property arises when using Sinc methods and it is now in the Poly-Sinc method.

5 Emden-Type Equations

In this section, we present various types of LE equations discussed in literatures [1, 2, 3, 6]. We classify these models as homogeneous $R(x) = 0$ and inhomogeneous LE equations $R(x) \neq 0$ in (1). We start with the discussion of homogeneous and then switch to inhomogeneous one.

5.1 Homogeneous Equations

In this section, we set $R(x) = 0$, $\alpha = 2$ in (1), and present various forms of $g(y)$ that have attracted attention, due to their significant applications. For example, an interesting form of $g(y)$ is $g(y) = e^y$ this model describes an isothermal gas sphere with a constant temperature setting m to be infinite. On the other hand, inserting $g(y) = e^{-y}$ into (1) creates a model that appears in the theory of thermionic currents thoroughly investigated by Richardson [3]. Furthermore, the function $g(y)$ appears in eight additional cases [2], namely, four trigonometric forms defined by, $g(y) = \pm\sin(y)$ and $g(y) = \pm\cos(x)$ and four hyperbolic functions, $g(y) = \pm\cosh(y)$ and $g(y) = \pm\sinh(x)$. A detailed discussion of the formulation of these models and the physical meaning of the solutions can be found in [1, 3].

Example 1. *Isothermal gas spheres equation*

$$\begin{aligned} y'' + \frac{2}{x}y' + e^y = 0,\ x > 0 \\ y(0) = 1,\ y'(0) = 0,\ y(1) = x_0 \end{aligned} \tag{26}$$

This model can be used to examine isothermal gas spheres, where the temperature remains constant and the index m is infinite [2].

Only one particular solution of (26) is known, namely, $y = \ln\left(\frac{2}{x^2}\right)$. This equation has been solved by using series expansion [2, 22] and by using Adomian decomposition [9, 23]. This solution can be given as:

$$\begin{aligned} y(x) = & - \frac{1}{6}x^2 + \frac{1}{5\times 5!}x^4 - \frac{8}{21\times 6!}x^6 \\ & + \frac{122}{81\times 8!}x^8 - \frac{61\times 67}{495\times 10!}x^{10}. \end{aligned} \tag{27}$$

We apply the Poly-Sinc algorithm to solve the isothermal gas spheres equation (26).

Figure (10) represents the approximate Poly-Sinc solution, dashed line, together with the analytic solution defined in (27), solid line.

Figure (11) displays the absolute error as a logarithmic plot using $N = 6$ in the Poly-Sinc calculations, means 13 Sinc points.

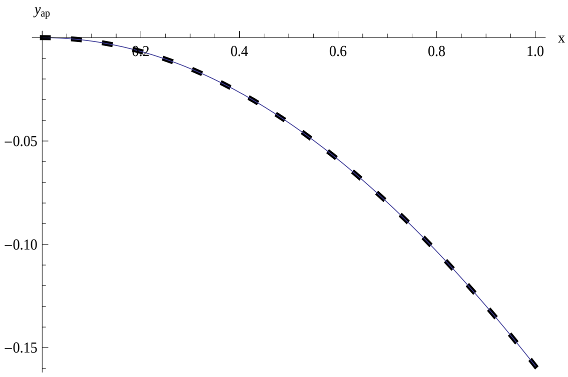

Figure 10: The exact and approximate solution of (26) with $N = 6$.

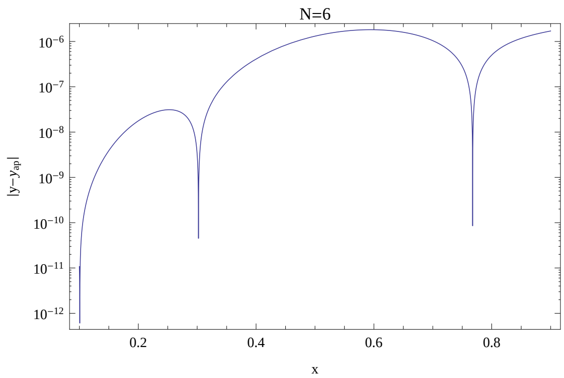

Figure 11: $|y - y_{ap}|$ for (26) $N = 6$.

Calculating the norm error ϵ based on the norm using (21) results to 10^{-6} for this small number of collocation points, 13 Sinc points.

Example 2. *Richardson's theory of thermionic currents*

Richardson [3] introduced a counter equation to (26) in which e^y is replaced by e^{-y}. The model is given by the nonlinear differential equation and its boundary conditions as:

$$y'' + \frac{2}{x}y' + e^{-y} = 0, \; x > 0 \tag{28}$$
$$y(0) = 1, \; y'(0) = 0, \; y(1) = x_0.$$

This model appears in Richardson's theory of thermionic currents when the density and electric force of an electron gas in the neighborhood of a hot body in thermal equilibrium is to be determined [3]. Equation (28) has a series solution based on using Adomain polynomial and can be given by [9]:

$$\begin{aligned} y(x) = & - \frac{1}{6}x^2 - \frac{1}{5\times 5!}x^4 - \frac{8}{21\times 6!}x^6 \\ & - \frac{122}{81\times 8!}x^8 - \frac{61\times 67}{495\times 10!}x^{10}. \end{aligned} \tag{29}$$

Figure (12) represents the analytic series solution defined in (29), solid line, and the approximate solution obtained from applying Poly-Sinc algorithm with 13 Sinc points, dashed line.

Figure (13) represents the absolute error between the approximate solution using the Poly-Sinc algorithm and the series solution in (29).

With this small number of Sinc collocation points, 13 Sinc points, we can get a norm error equal to $\epsilon = 1.96202 * 10^{-7}$.

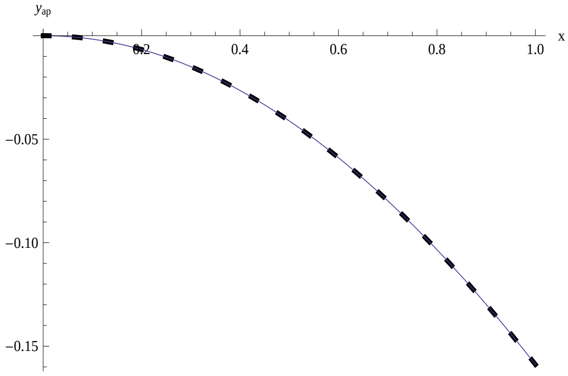

Figure 12: The exact and approximate solution of (28) with $N = 6$.

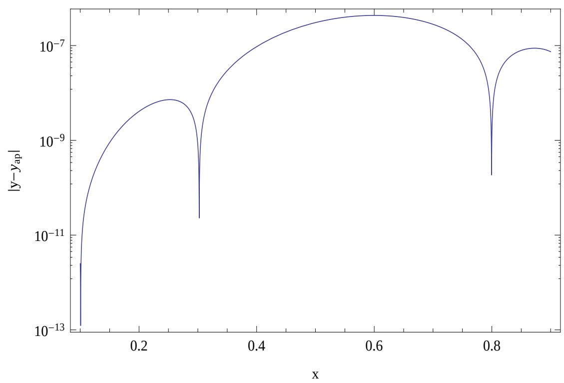

Figure 13: $|y - y_{ap}|$ for (28) $N = 6$.

Example 3. *Trigonometric and hyperbolic models*

In this part we will consider some other additional cases of $g(y)$ used in (21) that are discussed in [2], namely trigonometric and hyperbolic functions.

$$y'' + \frac{2}{x}y' + g(y) = 0, \; x > 0 \tag{30}$$
$$y(0) = 1, \; y'(0) = 0, \; y(1) = x_0,$$

Equation	Error
$y'' + \frac{2}{x}y' + \sin(y) = 0$	10^{-6}
$y'' + \frac{2}{x}y' + \sinh(y) = 0$	10^{-6}
$y'' + \frac{2}{x}y' + \cos(y) = 0$	10^{-7}
$y'' + \frac{2}{x}y' + \cosh(y) = 0$	10^{-12}

Table 2: The norm error for homogeneous LE equation- type for different $g(y)$ and $R(x)$ with $N = 6$.

where $g(y) = \sin(y), \cos(y), \sinh(y)$. For all of the cases of $g(y)$ table 2 represents the norm error between the approximate solution using Poly-Sinc algorithm and the analytic series solution discussed in [2].

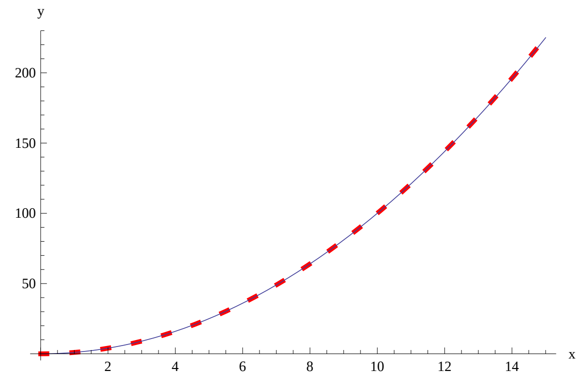

Figure 14: The exact and approximate solution of (31) with $N = 6$.

5.2 Inhomogeneous Equations

In this section a collection of inhomogeneous form of the LE type equations (1) is discussed. Some of these equations contain a singularity at $x = 0$ in the first derivative term only [10, 18, 24] and others have the singularity in both terms that are including the first derivative and the function $y(x)$ itself [10]. These types of equations have been solved using different techniques. Some of these techniques are differential transformation as in [24], Adomian decomposition as in [10] or, Taylor series [24]. Here we use the Poly-Sinc to solve examples of these inhomogeneous equations.

Example 4.

$$y'' + \frac{4}{x}y' + \frac{2}{x^2}y = 12, \; x > 0 \tag{31}$$
$$y(0) = 1, \; y'(0) = 0,$$

This equation is an inhomogeneous equation defined on the interval $(0, \infty)$ and has a singularity at $x = 0$.

In all of the previous examples, the equations were defined on finite intervals. The current example is defined on a semi-infinite interval, $(0, \infty)$. So a modifications in the definitions of the conformal mapping from the semi-infinite interval onto $\mathbb{R}$ is needed. This conformal map is,

$$\boldsymbol{\phi[\mathrm{x\_}]:=\mathrm{Log}\,[x]}$$

In this case its inverse conformal mapping can defined as,

$$\boldsymbol{\psi[\mathrm{x\_}]:=e^{x}}$$

The Sinc points on the interval $(0, \infty)$ is,

$$\boldsymbol{\mathrm{sp}[\mathrm{k\_}]:=\psi[kh]/.h \to \frac{\pi}{\sqrt{\mathrm{Nu}}};}$$

Now applying the Poly-Sinc algorithm to get an approximate solution of (31) on $(0, \infty)$.

Figure (14) represents the analytic solution of the inhomogeneous equation (31), solid line and the Poly-Sinc approximate solution with 13 Sinc points, dashed line.

Calculating the norm error ϵ using (21) to get $\epsilon = 8.5 * 10^{-4}$,

Table 3 summaries a collection of different inhomogeneous equations. Also it presents the norm error ϵ defined in (21) gained from comparing the approximate Poly-Sinc solution and the exact solution.

κ	$g(y)$	$R(x)$	y_{ex}	ϵ
2	y	x^2	x^3+x^2	10^{-11}
2	xy	$x^5-x^4+44x^2-30x$	x^4-x^3	10^{-15}
2	y	x^5+30x^2	x^5	10^{-12}
2	y^3	x^6+6	x^2	10^{-14}
6	$\frac{6y}{x^2}+y^2$	$20+x^4$	x^2	10^{-14}

Table 3: The obtained error using (21) for different inhomogeneous LE equations

6 Conclusion

The goal of this paper is to construct an approximation to the solution of nonlinear Lane- Emden type equations in finite and semi-infinite intervals. Lagrange approximation at Sinc data is proposed to provide an effective but simple way to improve the convergence of the solution by the collocation method. In some examples, a comparison is made between the exact and the approximate solution obtained by Poly-Sinc. Other examples of the examination compare between the series solutions and the approximation based on Poly-Sinc algorithm. It has been shown that the present work offers an effective approach for Lane-Emden type equations. Also we confirmed by logarithmic plots of the norm error, that this approach has an exponential convergence rate similar to the classical Sinc approximation. In total the Poly-Sinc method has three different properties: easy to compute and implement, exponential convergence, and handling all kind of singularities without modifications of the numerical scheme, which means that any solution can be represented to arbitrarily high accuracy for a small number of Sinc points.

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