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Inhomogeneous Plane Waves in Cubic Crystals subject to a Bias

LIVIA HARABAGIU

1. *Department of Mathematics,University of Bucharest, Bucharest,Romania,harabagiu_livia@yahoo.com*

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Abstract: In the present paper we investigate the condition of inhomogeneous plane waves propagation in cubic crystals subject to initial deformations and electric fields.The author obtains here the components of the electroacoustic tensor and the velocities of propagation as closed-form solutions.We show the influence of electrostrictive and piezoelectric effects on wave propagation in such media. We analyze the influence of the initial fields on the waves polarization in two main cases: (i)propagation in isotropic directional bivectors;(ii)propagation in case of polar anisotropic directional bivectors.

Key words: Inhomogeneous plane waves, cubic crystals, initial fields, isotropic/anisotropic directional bivectors.

1. Introduction

In the last years many authors have paid attention to the dynamics of electroelastic materials, which are subject to initial mechanical and electric fields. The basic equations of piezoelectric bodies for infinitesimal deformations and fields superposed on the initial deformations and electric fields, were given by Erigen and Maugin in their monograph [6].An alternative derivation of the equations of this type was obtained by Baesu and Soos in [1].The problem of waves propagation in elastic crystals and in piezoelectric crystals is presented in [10].In [16] the fundamental equations for piezoelectric crystals subject to initial fields have been re-established and important results concerning the dynamic and static local stability conditions of such media obtained.

In [11,12,13,14,15,16],Simionescu studied the propagation conditions of plane waves in cubic crystals subject to initial deformations and electric fields. In this paper we generalize the previous results, studying the problem of inhomogeneous plane waves in cubic crystals subject to initial electro-mechanical fields. We assume that the material is subject to initial electro-mechanical fields having small intensity. The propagation of elliptically polarized inhomogeneous plane waves has applications in many areas including Rayleigh, Love and Stoneley waves in classical linear elasticity theory. This concept may be found in paper [2] for anisotropic elasticity, in [3] for electromagnetism , in [4] for elastic materials with voids , in [5] for Hadamard material , or in [9] for viscoelasticity. In [17] we obtained the conditions of inhomogeneous plane wave propagation in monoclinic crystals subject to initial electromechanical fields. The concept of "bivector " is described in [8] .The algebra of bivectors is well established in [3,7,18].

In this paper we derive the decomposition of the propagation condition for particular isotropic /anisotropic directional bivectors , and we show that the specific coefficients are similar to the case of guided waves propagation in cubic crystals subject to a bias (see [16], to compare).

2. Basic equations

The basic equations of piezoelectric bodies for infinitesimal deformations and fields superposed on initial deformations and electric fields were given by

Eringen and Maugin in their monograph [6].An alternative derivation of these equations was obtained by Baesu, Fortune and Soos in [1].

We assume the material is an elastic dielectric, which is nonmagnetizable and conducts neither heat, nor electricity. We shall use the quasi-electrostatic approximation of the equations of balance. Furthermore, we assume that the elastic dielectric is linear and homogenous , that the initial homogenous electric field has small intensity. To describe this situation we use three different configurations:

- The reference configuration B_R in which at time t=0 the body is underformed and free of all fields;
- The initial configuration $\overset{\circ}{B}$ in which the body is deformed statically and carries the initial fields;
- The present(current) configuration B_t obtained from $\overset{\circ}{B}$ by applying time dependent incremental deformations and fields.In what follows, all the fields related to the initial configuration $\overset{\circ}{B}$ will be denoted by a superposed "$\circ$" .

The basic equations of the dynamic theory consist of the *following equations*(see paper [16] for details):

-the equations of motion :

$$\overset{\circ}{\rho}\ddot{u} = \mathrm{div}\Sigma\,; \qquad (1)$$

$$\mathrm{div}\Delta = 0 \quad \text{on} \quad \overset{\circ}{V}\,; \qquad (2)$$

- the equation of the electric field : (3)

$$e = -\mathrm{grad}\,\varphi\,;$$

- **the constitutive equations** :

$$\Sigma_{kl} = \overset{\circ}{\Omega}_{klmn}\, u_{m,n} + \overset{\circ}{\Lambda}_{mkl}\, \varphi_{,m}\,, \qquad (4)$$

$$\Delta_k = \overset{\circ}{\Lambda}_{kmn}\, u_{n,m} - \overset{\circ}{\varepsilon}_{kl}\, \varphi_{,l}\,, \qquad (5)$$

where $\overset{\circ}{\rho}$ is the mass density, **u** is the incremental displacement from $\overset{\circ}{B}$ to B_t , **Σ** is the incremental mechanical nominal stress, **Δ** is the incremental electric displacement vector , **φ** is the incremental electric potential, $\overset{\circ}{\Omega}$ is the instantaneous elasticity tensor , $\overset{\circ}{\Lambda}$ are the instantaneous coupling tensor and $\overset{\circ}{\varepsilon}$ are the instantaneous dielectric tensor.All incremental fields involved into the above equations depend on the spatial variable x and on time t .

The instantaneous coefficients can be expressed in terms of the classical moduli of the material and on the initial applied fields as follows :

$$\overset{\circ}{\Omega}klmn = \overset{\circ}{\Omega}nmlk = c_{klmn} + \overset{\circ}{S}kn\,\delta_{lm} - e_{kmn}\,\overset{\circ}{E}l - e_{nkl}\,\overset{\circ}{E}m - \eta_{kn}\,\overset{\circ}{E}l\,\overset{\circ}{E}m,$$

$$\overset{\circ}{\Lambda}mkl = e_{mkl} + \eta_{mk}\,\overset{\circ}{E}l,$$

(6)

$$\overset{\circ}{\varepsilon}kl = \overset{\circ}{\varepsilon}lk = \varepsilon_{lk} = \delta_{kl} + \eta_{kl},$$

where c_{klmn} are the components of the constant elasticity tensor, e_{kmn} are the components of the constant piezoelectric tensor , ε_{kl} are the components of the constant dielectric tensor , $\overset{\circ}{E}_i$ are the components of the initial applied electric field and

$\overset{\circ}{S}_{kn}$ are the components of the initial applied symmetric(Cauchy) stress tensor.

It is important to observe (to notice) that the previous material moduli have the following symmetry properties:

$$c_{klmn} = c_{lkmn} = c_{klnm} = c_{mnkl}, \quad (8)$$

$$e_{mkl} = e_{mlk}, \quad (9)$$

$$\varepsilon_{lk} = \varepsilon_{kl}. \quad (10)$$

From the previous field and constitutive equations we obtain *the following fundamental system* of four equations:

$$\overset{\circ}{\rho}\ddot{u}_l = \overset{\circ}{\Omega}_{klmn}\, u_{m,nk} + \overset{\circ}{\Lambda}_{mkl}\, \varphi_{,mk}, \quad (11)$$

$$\overset{\circ}{\Lambda}_{kmn}\, u_{n,mk} - \overset{\circ}{\varepsilon}_{kn}\, \varphi_{,nk} = 0,\ l = \overline{1,3}. \quad (12)$$

3. Inhomogeneous plane waves in piezoelectric crystals

In this section we deduce the equation for the slowness and for amplitude of inhomogeneous plane waves.For electromechanical problem (12) ,we define *the inhomogeneous plane wave* by :

$$u(x,t) = A\exp[i\omega(S\cdot x - t)], \quad (13)$$

$$\varphi(x,t) = \Phi\exp[i\omega(S\cdot x - t)],$$

where $A=A^{+}+iA^{-}$ is a complex vector defining *the mechanical amplitude* , Φ is *the electric amplitude* of the wave and $S=S^{+}+iS^{-}$ is a complex vector denoting *the slowness bivector* , ω is the *frequency* of the wave , which is a real parameter and $i=\sqrt{-1}$ is the complex unit.The superscripts "+" and "-" denote the real and imaginary parts of a complex quantity.

The real part of **u** is

$$u^{+} = \left[A^{+}\cos\omega\left(S^{+}\cdot x - t\right) - A^{-}\sin\omega\left(S^{+}\cdot x - t\right)\right]\exp\left(-\omega S^{-}\cdot x\right)$$

(14)

The planes $S^{+}\cdot x = \text{constant}$ are planes of constant phase, while $S^{-}\cdot x = \text{constant}$ are planes of constant amplitude.The relations (14) represent a train of elliptically polarized plane waves.The waves travel in the direction of the vector S^{+} , with the slowness $\left|S^{+}\right|$ and are attenuated in the direction of the vector S^{-} . The period is $2\pi/\omega$ .For any fixed position vector x, the displacement vector u^{+} describes an ellipse similar to the ellipse defined by the bivector A , namely the ellipse whose conjugate semi-diameters are $A^{+}\exp(-\omega S^{-}\cdot x)$ and $A^{-}\exp(-\omega S^{-}\cdot x)$. As t increases the sense in which the particle waves along the ellipse is from A^{+} to A^{-}.

Definition 1

A solution in the form (6)+(7) defines *an inhomogeneous plane wave* if the vector S^{-} is not parallel to the S^{+}.

We see that in the case of the inhomogeneous plane waves the planes of constant phase are different from the planes of constant amplitude.The *phase speed* is given by $V = \left|S^{+}\right|^{-1}$,while $\left|S^{-}\right|$ defines

the *attenuation coefficient*. In one particular case, when S^- is parallel to the S^+, we have *an attenuated homogeneous plane wave*(analyzed in Simionescu-Panait in [12],[13],[15] and [16]).

In order to solve the problem of inhomogeneous plane wave propagation in the above material , we use the *directional ellipse method* , due to Ph. Boulanger and M.Hayes in paper [3].So, the slowness bivectors may be written

$$\mathbf{S}=N\mathbf{C}, \qquad (15)$$

Where N is a complex number, $N = T\cdot e^{i\alpha}$, C is the directional bivector and it has the form

$$C=q\hat{m}+i\hat{n}, \text{ with } \hat{m}\cdot\hat{n}=0, \hat{m}\cdot\hat{n}=0 ,$$

$$|\hat{m}|=|\hat{n}|=1 \text{ and } q\geq 1. \qquad (16)$$

Thus, the slowness S,as well as the amplitude A and Φ are determined from the equations of motion (1)+(2).The main unknown data of the inhomogeneous plane wave propagation problem is the complex scalar slowness N.

We have :

$$\frac{\partial}{\partial t}(\cdot)=-i\omega(\cdot) \;;$$

$$\frac{\partial^2}{\partial t^2}(\cdot)=-\omega^2(\cdot) \;;$$

$$\frac{\partial u}{\partial x_i}=i\omega NC_i u \;; \qquad (17)$$

$$\frac{\partial \varphi}{\partial x_i}=-\omega NC_i\varphi \;;$$

$$\frac{\partial^2}{\partial x_i\partial x_j}(\cdot)=\frac{\partial^2}{\partial x_j\partial x_i}(\cdot)=-\omega^2N^2C_iC_j(\cdot);$$

$$\frac{\partial}{\partial x_l}\left(\frac{\partial^2 u}{\partial x_i\partial x_j}\right)=-\omega^2N^2C_iC_ju_l.$$

Inserting (13) into (12) gives:

$$-\overset{\circ}{\rho}\omega^2 a_l=-\omega^2\overset{\circ}{\Omega}_{klmn}N^2C_nC_ka_m-\overset{\circ}{\Lambda}_{mkl}\omega^2N^2C_mC_k\varphi,$$

(18)

$$\overset{\circ}{\Lambda}_{kmn}\omega^2N^2C_mC_ka_n+\overset{\circ}{\varepsilon}_{kn}\omega^2N^2C_nC_k\varphi=0,\; l,m=\overline{1,3}$$

.

From which we deduce:

$$\overset{\circ}{\Omega}_{klmn}N^2C_nC_ka_m+\overset{\circ}{\Lambda}_{mkl}N^2C_mC_k\varphi-\overset{\circ}{\rho}a_l=0,$$

(18)

$$\overset{\circ}{\Lambda}_{kmn}N^2C_mC_ka_n-\overset{\circ}{\varepsilon}_{kn}N^2C_nC_k\varphi=0,\; l,m=\overline{1,3}.$$

Taking $\varphi=a_4$ and $V=\frac{1}{N}$ we obtain:

$$\overset{\circ}{\Omega}_{klmn}C_nC_ka_m+\overset{\circ}{\Lambda}_{mkl}C_mC_ka_4-\overset{\circ}{\rho}V^2a_l=0,$$

(19)

$$\overset{\circ}{\Lambda}_{kmn}C_mC_ka_n-\overset{\circ}{\varepsilon}_{kn}C_nC_ka_4=0,\; l,m=\overline{1,3},$$

which gives:

$$\begin{pmatrix} \overset{\circ}{\Gamma}_{11}-\overset{\circ}{\rho}V^2 & \overset{\circ}{\Gamma}_{12} & \overset{\circ}{\Gamma}_{13} & \gamma_1 \\ \overset{\circ}{\Gamma}_{21} & \overset{\circ}{\Gamma}_{22}-\overset{\circ}{\rho}V^2 & \overset{\circ}{\Gamma}_{23} & \gamma_2 \\ \overset{\circ}{\Gamma}_{31} & \overset{\circ}{\Gamma}_{32} & \overset{\circ}{\Gamma}_{33}-\overset{\circ}{\rho}V^2 & \gamma_3 \\ \gamma_1 & \gamma_2 & \gamma_3 & -\overset{\circ}{\varepsilon} \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix}=0$$

(20)

where

$$\overset{\circ}{\Gamma}_{lm}=\overset{\circ}{\Omega}_{klmn}C_nC_k=\left(c_{klmn}+\overset{\circ}{S}_{kn}\delta_{lm}-e_{kmn}\overset{\circ}{E}_l-e_{nkl}\overset{\circ}{E}_m-\eta_{kn}\overset{\circ}{E}_l\overset{\circ}{E}_m\right)C_nC_k$$

$$\overset{\circ}{\gamma}_l=\overset{\circ}{\Lambda}_{mkl}C_mC_k=\left(e_{mkl}+\eta_{mk}\overset{\circ}{E}_l\right)C_mC_k,$$

(21)

$$\overset{\circ}{\varepsilon} = \overset{\circ}{\varepsilon}_{kn} C_n C_k = \left(\delta_{kn} + \eta_{kn}\right) C_n C_k .$$

Theorem 1

System (20) represents ***the propagation condition of the inhomogeneous plane waves*** inside the previous materials and is equivalent to :

$$\begin{pmatrix} \overset{\circ}{Q}_{lm} & \overset{\circ}{Q}_{l4} \\ \overset{\circ}{Q}_{4m} & \overset{\circ}{Q}_{44} \end{pmatrix} \begin{pmatrix} a_m \\ a_4 \end{pmatrix} = 0, l, m = \overline{1,3} \quad (22) \quad ,$$

where Q is the electroacoustic tensor and has the following components:

$$\overset{\circ}{Q}_{lm} = N^2 \overset{\circ}{\Omega}_{klmn} C_n C_k - \overset{\circ}{\rho} \delta_{lm},$$

$$\overset{\circ}{Q}_{l4} = N^2 \overset{\circ}{\Lambda}_{mkl} C_m C_k, \qquad (23)$$

$$\overset{\circ}{Q}_{4m} = -N^2 \overset{\circ}{\varepsilon}_{kn} C_n C_k .$$

Remark 1:
For the general anisotropy, the tensor Q is symmetrical.

Theorem 2

Linear algebraic system admits a non-zero solution (a_1, a_2, a_3, a_4) if and only if N satisfies the following algebraic equation:

$$\det \begin{pmatrix} \overset{\circ}{Q}_{lm} & \overset{\circ}{Q}_{l4} \\ \overset{\circ}{Q}_{4m} & \overset{\circ}{Q}_{44} \end{pmatrix} = 0, l, m = \overline{1,3} \ . \ (24)$$

Remark 2:
For any prescribed directional bivector $C = q\hat{m} + i\hat{n}$, the values of the complex number N are obtained by solving the generalized form (24) of the secular equation , while the corresponding non-zero solutions (a_1, a_2, a_3, a_4) are obtained by solving the corresponding system (22).

4.Inhomogeneous plane waves propagation in cubic crystals

It is known that , in the particular case of a cubic crystal , the elasticity tensor contains three independent constants (see [10]).Using Voight's convention we have:

$$c = \begin{pmatrix} c_{11} & c_{12} & c_{12} & 0 & 0 & 0 \\ c_{12} & c_{11} & c_{12} & 0 & 0 & 0 \\ c_{12} & c_{12} & c_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{44} \end{pmatrix} \quad (25)$$

Among the five symmetry classes belonging to the cubic system , only $\overline{43m}$ and 23 classes exhibit the piezoelectric effect , for the others (i.e. m3m , 432) the piezoelectric effect is absent. Similarly, the piezoelectric tensor contains one constant :

$$e = \begin{pmatrix} 0 & 0 & 0 & e_{14} & 0 & 0 \\ 0 & 0 & 0 & 0 & e_{14} & 0 \\ 0 & 0 & 0 & 0 & 0 & e_{14} \end{pmatrix} \quad (26) \quad ,$$

while the dielectric tensor has one constant , for all five symmetry classes:

$$\eta = \begin{pmatrix} \eta & 0 & 0 \\ 0 & \eta & 0 \\ 0 & 0 & \eta \end{pmatrix} \quad . \qquad (27)$$

From relations (17) it follows that the acoustic tensor $\overset{\circ}{Q}$ has the following components :

$$\overset{\circ}{Q}_{11} = (c_{11} + \overset{\circ}{S}11) N^2 C_1^2 + (c_{44} + \overset{\circ}{S}22) N^2 C_2^2 + (c_{44} + \overset{\circ}{S}33) N^2 C_3^2 + 2\overset{\circ}{S}12\, N^2 C_1 C_2 +$$
$$2\overset{\circ}{S}13\, N^2 C_1 C_3 + 2\overset{\circ}{S}23\, N^2 C_2 C_3 - 4e_{14} \overset{\circ}{E}1\, N^2 C_2 C_3 - \eta\left(C_1^2 + C_2^2 + C_3^2\right) N^2 \overset{\circ}{E}1^2 - \overset{\circ}{\rho},$$

$$\overset{\circ}{Q}_{12} = \overset{\circ}{Q}_{21} = (c_{12} + c_{44})N^2 C_1 C_2 - 2e_{14}(\overset{\circ}{E}_2 C_2 + \overset{\circ}{E}_1 C_1)N^2 C_3 - \eta\left(C_1^2 + C_2^2 + C_3^2\right)N^2 \overset{\circ}{E}_1 \overset{\circ}{E}_2,$$

$$\overset{\circ}{Q}_{13} = \overset{\circ}{Q}_{31} = (c_{12} + c_{44})N^2 C_1 C_3 - 2e_{14}(\overset{\circ}{E}_3 C_3 + \overset{\circ}{E}_1 C_1)N^2 C_2 - \eta\left(C_1^2 + C_2^2 + C_3^2\right)N^2 \overset{\circ}{E}_1 \overset{\circ}{E}_3,$$

$$\overset{\circ}{Q}_{22} = (c_{44} + \overset{\circ}{S}_{11})N^2 C_1^2 + (c_{11} + \overset{\circ}{S}_{22})N^2 C_2^2 + (c_{44} + \overset{\circ}{S}_{33})N^2 C_3^2 + 2\overset{\circ}{S}_{12} N^2 C_1 C_2 + 2\overset{\circ}{S}_{13} N^2 C_1 C_3 + 2\overset{\circ}{S}_{23} N^2 C_2 C_3 - 4e_{14} \overset{\circ}{E}_2 N^2 C_1 C_3 - \eta\left(C_1^2 + C_2^2 + C_3^2\right)N^2 \overset{\circ}{E}{}_2^2 - \overset{\circ}{\rho},$$

$$\overset{\circ}{Q}_{23} = \overset{\circ}{Q}_{32} = (c_{12} + c_{44})N^2 C_2 C_3 - 2e_{14}(\overset{\circ}{E}_3 C_3 + \overset{\circ}{E}_2 C_2) N^2 C_1 - \eta\left(C_1^2 + C_2^2 + C_3^2\right)N^2 \overset{\circ}{E}_2 \overset{\circ}{E}_3,$$

(28)

$$\overset{\circ}{Q}_{33} = (c_{44} + \overset{\circ}{S}_{11})N^2 C_1^2 + (c_{44} + \overset{\circ}{S}_{22})N^2 C_2^2 + (c_{11} + \overset{\circ}{S}_{33})N^2 C_3^2 + 2\overset{\circ}{S}_{12} N^2 C_1 C_2 + 2\overset{\circ}{S}_{13} N^2 C_1 C_3 + 2\overset{\circ}{S}_{23} N^2 C_2 C_3 - 4e_{14} \overset{\circ}{E}_3 N^2 C_1 C_2 - \eta\left(C_1^2 + C_2^2 + C_3^2\right)N^2 \overset{\circ}{E}{}_3^2 - \overset{\circ}{\rho},$$

$$\overset{\circ}{Q}_{14} = \overset{\circ}{Q}_{41} = 2e_{14} N^2 C_2 C_3 + \eta\left(C_1^2 + C_2^2 + C_3^2\right)N^2 \overset{\circ}{E}_1,$$

$$\overset{\circ}{Q}_{24} = \overset{\circ}{Q}_{42} = 2e_{14} N^2 C_1 C_3 + \eta\left(C_1^2 + C_2^2 + C_3^2\right)N^2 \overset{\circ}{E}_2,$$

$$\overset{\circ}{Q}_{34} = \overset{\circ}{Q}_{43} = 2e_{14} N^2 C_2 C_1 + \eta\left(C_1^2 + C_2^2 + C_3^2\right)N^2 \overset{\circ}{E}_3,$$

$$\overset{\circ}{Q}_{44} = -\left(1 + \eta\right)\left(C_1^2 + C_2^2 + C_3^2\right)N^2.$$

4.1.Longitudinal waves

Definition 3

A bivector C is said to be ***isotropic*** if $C \cdot C = 0$.(29)

Remark 3

We consider the particular case of isotropic directional bivectors and we can choose C= $\mathbf{i} + i\mathbf{j}$ =(1,i,0) , where $\{0, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ represents an orthornorrmal basis of three dimensional Euclidian space and i is the complex unit.Here the inhomogeneous wave is circularly polarized in a plane normal to the axis x_3 .In this case, the corresponding amplitude and slowness bivectors are parallelled that is

$$A \times S = 0 \qquad (30)$$

We take

$$A = \alpha S = \left(\alpha N, \alpha N i, 0\right) \qquad (31) \quad ,$$

where α is a complex number.We have
C_1C_2=i,
C_1C_3=C_2C_3=0,

$$C_1^2 = 1, C_2^2 = -1, C_3^2 = 0, C_1^2 + C_2^2 + C_3^2 = 0. \quad (32)$$

From (28) , the components of the electroacoustic tensor are formed:

$$\overset{\circ}{Q}_{11} = (c_{11} + \overset{\circ}{S}_{11} - c_{44} - \overset{\circ}{S}_{22})N^2 + 2\overset{\circ}{S}_{12} N^2 i - \overset{\circ}{\rho},$$

$$\overset{\circ}{Q}_{12} = \overset{\circ}{Q}_{21} = (c_{12} + c_{44})N^2 i,$$

$$\overset{\circ}{Q}_{13} = \overset{\circ}{Q}_{31} = -2e_{14} N^2 i \overset{\circ}{E}_1, \qquad (33)$$

$$\overset{\circ}{Q}_{22} = (c_{44} + \overset{\circ}{S}_{11} - c_{11} - \overset{\circ}{S}_{22})N^2 + 2\overset{\circ}{S}_{12} N^2 i - \overset{\circ}{\rho},$$

$$\overset{\circ}{Q}_{23} = \overset{\circ}{Q}_{32} = -2e_{14} N^2 i \overset{\circ}{E}_2,$$

$$\overset{\circ}{Q}_{33} = (\overset{\circ}{S}_{11} - \overset{\circ}{S}_{22})N^2 + 2\overset{\circ}{S}_{12} N^2 i - 4e_{14} \overset{\circ}{E}_3 N^2 i - \overset{\circ}{\rho},$$

$$\overset{\circ}{Q}_{14} = \overset{\circ}{Q}_{24} = \overset{\circ}{Q}_{44} = 0,$$

$$\overset{\circ}{Q}_{34} = 2e_{14}N^2 i.$$

Thus , the equation (24) reduces to solving two conditions:

a) The first equation is

$$\overset{\circ}{Q}_{11}\overset{\circ}{Q}_{22} - \overset{\circ}{Q}{}_{12}^{2} = 0 \qquad (34)$$

and defines *a non-piezoelectric wave ,polarized in the plane x_1x_2, which depends on the initial stress field, only.This wave is marked* $\overset{\circ}{P}_2$.

b) The second equation is

$$\overset{\circ}{Q}_{34} = 0 \qquad (35)$$

has a *piezoelectric wave.This wave is noted* $\overset{\circ}{\overline{TH}}$.

In the first case, we noted

$$V = \frac{1}{N} \text{ and } \overset{\circ}{\rho} V^2 = x \quad . \quad (36)$$

The relation (34) becomes

$$x^2 - \left(\overset{\circ}{\Gamma}_{11} + \overset{\circ}{\Gamma}_{22}\right)x + \left(\overset{\circ}{\Gamma}_{11}\overset{\circ}{\Gamma}_{22} - \overset{\circ}{\Gamma}{}_{12}^{2}\right) = 0 \quad , \quad (37)$$

where

$$\overset{\circ}{\Gamma}_{11} = (c_{11} + \overset{\circ}{S}_{11} - c_{44} - \overset{\circ}{S}_{22}) + 2\overset{\circ}{S}_{12}\, i,$$

$$\overset{\circ}{\Gamma}_{12} = (c_{12} + c_{44})i, \qquad (38)$$

$$\overset{\circ}{\Gamma}_{22} = (c_{44} + \overset{\circ}{S}_{11} - c_{11} - \overset{\circ}{S}_{22}) + 2\overset{\circ}{S}_{12}\, i.$$

We have $\Delta_x = 4(c_{11} + c_{12})(c_{11} - c_{12} - 2c_{44})$ .In the case of cubic crystals, we have $\Delta_x < 0$. Then

$$x_{1,2} = \overset{\circ}{S}_{11} - \overset{\circ}{S}_{22} + 2\overset{\circ}{S}_{12}\, i \pm i\sqrt{(2c_{44} + c_{12} - c_{11})(c_{11} + c_{12})},$$

(39)

$$N_{1,2}^2 = \frac{\overset{\circ}{\rho}}{\overset{\circ}{S}_{11} - \overset{\circ}{S}_{22} + 2\overset{\circ}{S}_{12}\, i \pm i\sqrt{(2c_{44} + c_{12} - c_{11})(c_{11} + c_{12})}}.$$

If

$$N_1^2 = \frac{\overset{\circ}{\rho}}{\overset{\circ}{S}_{11} - \overset{\circ}{S}_{22} + 2\overset{\circ}{S}_{12} + i\sqrt{(2c_{44} + c_{12} - c_{11})(c_{11} + c_{12})}}$$

then we have $N_1 = \mathrm{Re}(N_1) + i\,\mathrm{Im}(N_1)$, (40)

where

$$\mathrm{Re}(N_1)_{1,2} = \pm\sqrt{\frac{\overset{o}{\rho}\left(\overset{o}{S}_{11} - \overset{o}{S}_{22}\right) + \sqrt{\overset{o}{\rho}\left(\overset{o}{S}_{11} - \overset{o}{S}_{22}\right)^2 + \left[2\overset{o}{S}_{12} + \sqrt{(2c_{44} + c_{12} - c_{11})(c_{11} + c_{22})}\right]^2}}{2\left(\overset{o}{S}_{11} - \overset{o}{S}_{22}\right)^2 + \left[2\overset{o}{S}_{12} + \sqrt{(2c_{44} + c_{12} - c_{11})(c_{11} + c_{22})}\right]^2}}$$

$$\mathrm{Im}(N_1)_{1,2} = \pm\sqrt{\frac{-\overset{\circ}{\rho}\left(\overset{\circ}{S}_{11} - \overset{\circ}{S}_{22}\right) + \sqrt{\overset{\circ}{\rho}^2\left(\overset{\circ}{S}_{11} - \overset{\circ}{S}_{22}\right)^2 + \left[2\overset{\circ}{S}_{12} + \sqrt{(2c_{44} + c_{12} - c_{11})(c_{11} + c_{22})}\right]^2}}{2\left(\overset{\circ}{S}_{11} - \overset{\circ}{S}_{22}\right)^2 + \left[2\overset{\circ}{S}_{12} + \sqrt{(2c_{44} + c_{12} - c_{11})(c_{11} + c_{22})}\right]^2}}$$

We obtained :

$$N_1 = \mathrm{Re}(N_1) + i\,\mathrm{Im}(N_1) = \sqrt{\frac{\theta + \beta}{\gamma}} + i\sqrt{\frac{-\theta + \beta}{\gamma}},$$

$$\theta = -\overset{\circ}{\rho}\left(\overset{\circ}{S}_{11} - \overset{\circ}{S}_{22}\right),$$

$$\beta = \sqrt{\overset{\circ}{\rho}^2\left(\overset{\circ}{S}_{11} - \overset{\circ}{S}_{22}\right)^2 + \left[2\overset{\circ}{S}_{12} + \sqrt{(2c_{44} + c_{12} - c_{11})(c_{11} + c_{22})}\right]^2},$$

$$\gamma = 2\left(\overset{\circ}{S}_{11} - \overset{\circ}{S}_{22}\right)^2 + \left[2\overset{\circ}{S}_{12} + \sqrt{(2c_{44} + c_{12} - c_{11})(c_{11} + c_{22})}\right]^2$$

$$S_1 = \left(\mathrm{Re}(N_1) + i\,\mathrm{Im}(N_1), \mathrm{Re}(N_1) - i\,\mathrm{Im}(N_1), 0\right),$$

$$A_1 = (\alpha N_1, \alpha N_1 i, 0),$$

$$\Phi_1 = 0 \quad . (41)$$

and

$$N_2 = \mathrm{Re}(N_2) + i\,\mathrm{Im}(N_2) = -\sqrt{\frac{\theta+\beta}{\gamma}} - i\sqrt{\frac{-\theta+\beta}{\gamma}},$$

$$\theta = -\overset{\circ}{\rho}\left(\overset{\circ}{S}_{11} - \overset{\circ}{S}_{22}\right),$$

$$\beta = \sqrt{\overset{\circ}{\rho}^2\left(\overset{\circ}{S}_{11} - \overset{\circ}{S}_{22}\right)^2 + \left[2\overset{\circ}{S}_{12} + \sqrt{(2c_{44} + c_{12} - c_{11})(c_{11} + c_{22})}\right]^2},$$

$$\gamma = 2\left(\overset{\circ}{S}_{11} - \overset{\circ}{S}_{22}\right)^2 + \left[2\overset{\circ}{S}_{12} + \sqrt{(2c_{44} + c_{12} - c_{11})(c_{11} + c_{22})}\right]^2$$

$$S_2 = \left(\mathrm{Re}(N_2) + i\,\mathrm{Im}(N_2), \mathrm{Re}(N_2) - i\,\mathrm{Im}(N_2), 0\right), \quad (42)$$

$$A_2 = (\alpha N_2, \alpha N_2 i, 0),$$

$$\Phi_2 = 0.$$

Therefore, the general solution of the system (12) is

$$u(x,t) = u_1(x,t) + u_2(x,t) \quad ,$$

where

$$u_1(x,t) = A_1 \exp[i\omega(S_1 \cdot x - t)] = (\alpha N_1 + \alpha N_1 i)\exp[i\omega(N_1 \cdot x_1 + N_1 \cdot x_2 + N_1 \cdot x_3 - t)], \quad (43)$$

$$u_2(x,t) = A_2 \exp[i\omega(S_2 \cdot x - t)] = (\alpha N_2 + \alpha N_2 i)\exp[i\omega(N_2 \cdot x_1 + N_2 \cdot x_2 + N_2 \cdot x_3 - t)]. \quad (44)$$

*4.2.**Transverse waves***

In this case , the corresponding and slowness bivectors are orthogonal , that is :

$$A \cdot S = 0 \quad . \qquad (45)$$

Remark 4

The plane of constant amplitude is orthogonal to the plane of constant phase ($S^+ \cdot S^- = 0$).The bivector **C** may not be isotropic.As in [2],[3],[4] , we choose

$$A = \delta C_\perp + \gamma \hat{m} \times \hat{n}, \qquad (46)$$

where

$$C_\perp = q^{-1}\hat{m} + i\hat{n}$$

is the reciprocal of the bivector C and δ and γ are arbitrary scalars .

Thus , in the second case we choose an anisotropic directional bivector

$$C = (C_1, C_2, 0) \quad , \qquad (47)$$

with

$$C_1 = \cos\alpha + i\sin\alpha,$$

$$C_2 = \cos\alpha - i\sin\alpha, \alpha \in [0, 2\pi).$$

This inhomogeneous wave is elliptically polarized in the plane normal to the axis x_3,except for the particular directions $\alpha \in \left\{\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}\right\}$. In this case , the wave is circularly polarized.

We have :

$C_1C_2=1$, $C_1C_3=C_2C_3=0$,

$$\begin{aligned} C_1^2 &= \cos 2\alpha + i\sin 2\alpha, \\ C_2^2 &= \cos 2\alpha - i\sin 2\alpha, \\ C_3^2 &= 0, \\ C_1^2 + C_2^2 + C_3^2 &= 2\cos 2\alpha. \end{aligned} \qquad (48)$$

From (28) , the components of the electroacoustic tensor are formed:

$$\overset{\circ}{Q}_{11} = (c_{11} + \overset{\circ}{S}_{11} + c_{44} + \overset{\circ}{S}_{22} - 2\eta \overset{\circ}{E}_1^2)N^2 \cos 2\alpha + 2\overset{\circ}{S}_{12} N^2 - \overset{\circ}{\rho} + i\left(c_{11} + \overset{\circ}{S}_{11} - c_{44} - \overset{\circ}{S}_{22}\right)N^2 \sin 2\alpha,$$

$$\overset{\circ}{Q}_{12} = \overset{\circ}{Q}_{21} = (c_{12} + c_{44})N^2 - 2\eta\cos 2\alpha N^2 \overset{\circ}{E}_1 \overset{\circ}{E}_2,$$

$$\overset{\circ}{Q}_{13} = \overset{\circ}{Q}_{31} = -2e_{14}N^2 \overset{\circ}{E}_1 - 2\eta\cos 2\alpha N^2 \overset{\circ}{E}_1 \overset{\circ}{E}_3, \quad (49)$$

$$\overset{\circ}{Q}_{22} = (c_{11} + \overset{\circ}{S}11 + c_{44} + \overset{\circ}{S}22 - 2\eta \overset{\circ}{E}2^2)N^2 \cos 2\alpha + 2\overset{\circ}{S}12\, N^2 - \overset{\circ}{\rho} + i\left(-c_{11} + \overset{\circ}{S}11 + c_{44} - \overset{\circ}{S}22\right)N^2 \sin 2\alpha,$$

$$\overset{\circ}{Q}_{23} = \overset{\circ}{Q}_{32} = -2e_{14}N^2 \overset{\circ}{E}_2 - 2\eta\cos 2\alpha N^2 \overset{\circ}{E}_2 \overset{\circ}{E}_3,$$

$$\overset{\circ}{Q}_{33} = (2c_{44} + \overset{\circ}{S}11 + \overset{\circ}{S}22 - 2\eta \overset{\circ}{E}3^2)N^2 \cos 2\alpha + 2\overset{\circ}{S}12\, N^2 - 4e_{14} \overset{\circ}{E}3 - \overset{\circ}{\rho} + i\left(\overset{\circ}{S}11 - \overset{\circ}{S}22\right)N^2 \sin 2\alpha,$$

$$\overset{\circ}{Q}_{14} = \overset{\circ}{Q}_{41} = 2\eta\cos 2\alpha N^2 \overset{\circ}{E}_1,$$

$$\overset{\circ}{Q}_{24} = \overset{\circ}{Q}_{42} = 2\eta\cos 2\alpha N^2 \overset{\circ}{E}_2,$$

$$\overset{\circ}{Q}_{34} = \overset{\circ}{Q}_{43} = 2e_{14}N^2 + 2\eta\cos 2\alpha N^2 \overset{\circ}{E}_3,$$

$$\overset{\circ}{Q}_{44} = -2N^2(1+\eta)\cos 2\alpha.$$

4.2.1.

If $\overset{\circ}{E}_1 = \overset{\circ}{E}_2 = 0$,we obtain :

$$\overset{\circ}{Q}_{11} = (c_{11} + \overset{\circ}{S}11 + c_{44} + \overset{\circ}{S}22)N^2 \cos 2\alpha + 2\overset{\circ}{S}12\, N^2 - \overset{\circ}{\rho} + i\left(c_{11} + \overset{\circ}{S}11 - c_{44} - \overset{\circ}{S}22\right)N^2 \sin 2\alpha,$$

$$\overset{\circ}{Q}_{12} = \overset{\circ}{Q}_{21} = (c_{12} + c_{44})N^2, \quad \overset{\circ}{Q}_{13} = \overset{\circ}{Q}_{31} = 0,$$

$$\overset{\circ}{Q}_{22} = (c_{11} + \overset{\circ}{S}11 + c_{44} + \overset{\circ}{S}22)N^2 \cos 2\alpha + 2\overset{\circ}{S}12\, N^2 - \overset{\circ}{\rho} + i\left(-c_{11} + \overset{\circ}{S}11 + c_{44} - \overset{\circ}{S}22\right)N^2 \sin 2\alpha,$$

$$\overset{\circ}{Q}_{23} = \overset{\circ}{Q}_{32} = 0,$$

$$\overset{\circ}{Q}_{33} = (2c_{44} + \overset{\circ}{S}11 + \overset{\circ}{S}22 - 2\eta \overset{\circ}{E}3^2)N^2 \cos 2\alpha + 2\overset{\circ}{S}12\, N^2 - 4e_{14} \overset{\circ}{E}3 - \overset{\circ}{\rho} + i\left(\overset{\circ}{S}11 - \overset{\circ}{S}22\right)N^2 \sin 2\alpha, \quad (50)$$

$$\overset{\circ}{Q}_{14} = \overset{\circ}{Q}_{41} = 0,$$

$$\overset{\circ}{Q}_{24} = \overset{\circ}{Q}_{42} = 0,$$

$$\overset{\circ}{Q}_{34} = \overset{\circ}{Q}_{43} = 2e_{14}N^2 + 2\eta\cos 2\alpha N^2 \overset{\circ}{E}_3,$$

$$\overset{\circ}{Q}_{44} = -2N^2(1+\eta)\cos 2\alpha.$$

Then, system (20) reduces to two independent subsystems , as follows:

- The first subsystem

$$\begin{pmatrix} \overset{\circ}{Q}_{11} & \overset{\circ}{Q}_{12} \\ \overset{\circ}{Q}_{12} & \overset{\circ}{Q}_{22} \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0 \qquad (51)$$

defines *a non-piezoelectric wave , polarized in the plane x_1x_2 , which depends on the initial stress field,only.It corresponds to* $\overset{\circ}{P}_2$ *guided wave.*

- The second subsystem

$$\begin{pmatrix} \overset{\circ}{Q}_{33} & \overset{\circ}{Q}_{34} \\ \overset{\circ}{Q}_{34} & \overset{\circ}{Q}_{44} \end{pmatrix}\begin{pmatrix} a_3 \\ a_4 \end{pmatrix} = 0 \qquad (52)$$

has a solution *a transverse-horizontal wave ,with polarization after axis x_3,which is piezoelectric and electrostrictive active , and depends on the initial mechanical and electrical fields.This wave is linked to* $\overset{\circ}{\overline{TH}}$ *guided wave.*

4.2.2

If $\overset{\circ}{E}_3 = 0$,we obtain :

$$\overset{\circ}{Q}_{11} = (c_{11} + \overset{\circ}{S}11 + c_{44} + \overset{\circ}{S}22 - 2\eta \overset{\circ}{E}1^2)N^2 \cos 2\alpha + 2\overset{\circ}{S}12\, N^2 - \overset{\circ}{\rho} + i\left(c_{11} + \overset{\circ}{S}11 - c_{44} - \overset{\circ}{S}22\right)N^2 \sin 2\alpha,$$

$$\overset{\circ}{Q}_{12} = \overset{\circ}{Q}_{21} = (c_{12} + c_{44})N^2 - 2\eta\cos 2\alpha N^2 \overset{\circ}{E}_1 \overset{\circ}{E}_2,$$

$$\overset{\circ}{Q}_{13} = \overset{\circ}{Q}_{31} = -2e_{14}N^2\,\overset{\circ}{E}_1,$$

$$\overset{\circ}{Q}_{22} = (c_{11} + \overset{\circ}{S}11 + c_{44} + \overset{\circ}{S}22 - 2\eta\,\overset{\circ}{E}2^2)N^2\cos 2\alpha + 2\overset{\circ}{S}12\,N^2 - \overset{\circ}{\rho} + i\left(-c_{11} + \overset{\circ}{S}11 + c_{44} - \overset{\circ}{S}22\right)N^2\sin 2\alpha,$$

$$\overset{\circ}{Q}_{23} = \overset{\circ}{Q}_{32} = -2e_{14}N^2\,\overset{\circ}{E}_2,$$

$$\overset{\circ}{Q}_{33} = (2c_{44} + \overset{\circ}{S}11 + \overset{\circ}{S}22)N^2\cos 2\alpha + 2\overset{\circ}{S}12\,N^2 - \overset{\circ}{\rho} + i\left(\overset{\circ}{S}11 - \overset{\circ}{S}22\right)N^2\sin 2\alpha, \qquad (53)$$

$$\overset{\circ}{Q}_{14} = \overset{\circ}{Q}_{41} = 2\eta\cos 2\alpha N^2\,\overset{\circ}{E}_1,$$

$$\overset{\circ}{Q}_{24} = \overset{\circ}{Q}_{42} = 2\eta\cos 2\alpha N^2\,\overset{\circ}{E}_2,$$

$$\overset{\circ}{Q}_{34} = \overset{\circ}{Q}_{43} = 2e_{14}N^2,$$

$$\overset{\circ}{Q}_{44} = -2N^2(1+\eta)\cos 2\alpha.$$

The system (20) reduces to two independent subsystems , as follows :

- The first subsystem has the form :

$$\begin{pmatrix} \overset{\circ}{Q}_{11} & \overset{\circ}{Q}_{12} & \overset{\circ}{Q}_{14} \\ \overset{\circ}{Q}_{12} & \overset{\circ}{Q}_{22} & \overset{\circ}{Q}_{24} \\ \overset{\circ}{Q}_{14} & \overset{\circ}{Q}_{24} & \overset{\circ}{Q}_{44} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_4 \end{pmatrix} = 0 \qquad (54)$$

defines *an inhomogeneous plane wave, polarized into the plane x_1x_2,associated with the electric field , providing piezoelectric and electrostrictive effects , and depending on the initial stress and electric fields .*

It corresponds to $\overset{\circ}{\overline{P}}_2$ *wave from guided wave propagation problem.*

- The second subsystem is reduced to a single equation , as follows :

$$\overset{\circ}{Q}_{33}\,a_3 = 0 \qquad (55)$$

defines *a transverse-horizontal wave, with polarization after the axis x_3,non-piezoelectric and influenced by the initial stress field, only.It corresponds to* $\overset{\circ}{T}H$ *wave form the problem of guided wave propagation.*

5.Conclusions

In our paper we obtained the conditions of inhomogeneous plane wave propagation in cubic crystals subject to initial electromechanical fields.For particular isotropic and anisotropic directional bivectors we derive decomposition of the propagation condition.We show that the specific coefficients are similar to guided waves propagation in monoclinic crystals subject to a bias.

We analyzed the influence of the initial mechanical and electric fields on the wave propagation.Considering the particular cases of longitudinal inhomogeneous waves, we find *the velocities of propagation* and the *polarization of the waves* , via *the electrostrictive effect*.

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